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ABSTRACT

This is part one of a two-part manual for teachers using SMSG junior high school text materials. A chapter-by-chapter commentary on the text is given as well as answers to all the exercises. Chapter topics include: (1) number line and coordinates; (2) equations; (3) scientific notation; (4) applications of percent; and (5) congruence and the Pythagorean Property. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

MATHEMATICS FOR JUNIOR HIGH SCHOOL

Commentary for Teachers

VOLUME II (Part 1)

2

MATHEMATICS FOR JUNIOR HIGH SCHOOL

Commentary for Teachers

Volume II (Part 1)

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This volume was prepared at a writing session held at the University of Michigan, June 15–August 7, 1959. It is based, in part, on material prepared at the first SMSG writing session, held at Yale University in the summer of 1958. The members of the Ann Arbor writing team were:

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FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group. This Study Group includes college and university mathematicians, high school teachers of mathematics, experts in education, and representatives of science and technology. The general objective of the Study Group is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by the School Mathematics Study Group was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of high school textbooks which would illustrate such an improved curriculum. This textbook is based upon 14 experimental units which comprised a first product of this project.

The professional mathematicians in the Study Group believe that the mathematics presented in this text is important for all well-educated citizens in our society to know and that it is also important for the pre-college student to learn in preparation for advanced work in the field. At the same time, the high school teachers in the Study Group believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material presented will have a familiar note to it, but the flavor of presentation, the point of view, as it were, will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead a student to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of the commercial textbooks of the future. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

TABLE OF CONTENTS

PREFACE

Fourteen experimental units for use in the seventh and eighth grades were written in the summer of 1958 and tried out by approximately 100 teachers in 12 centers in various parts of the country in the school year 1958-59. Some of the chapters of this book are quite similar to the corresponding experimental units, and in others there are some important changes. Several of the chapters are entirely new. The changes and additions are based both on the teachers' comments on their experience in teaching the experimental units, and on knowledge of modern needs. The materials are also carefully chosen to provide adequate preparation for the text materials prepared for use in grades 9 through 12 by SMSG.

Big ideas of junior high school mathematics, emphasized in this text are: structure of arithmetic from an algebraic viewpoint; the real number system as a progressing development; metric and non-metric relations in geometry. Throughout the materials these ideas are associated with their applications. Important at this level are experience with and appreciation of abstract concepts, the role of definition, development of precise vocabulary and thought, experimentation, and proof. Substantial progress can be made on these concepts in the junior high school.

Mathematics is fascinating to many persons because of its opportunities for creation and discovery as well as for its utility. It is continuously and rapidly growing under the prodding of both intellectual curiosity and practical applications. Even junior high school students may formulate mathematical questions and conjectures which they can test and perhaps settle; they can develop systematic attacks on mathematical problems whether or not the problems have routine or immediately determinable solutions. Recognition of these important factors has played a considerable part in selection of content and method in this text.

We firmly believe mathematics can and should be studied with success and enjoyment. It is our hope that this text may greatly assist all teachers who use it to achieve this highly desirable goal.

8

Unit I

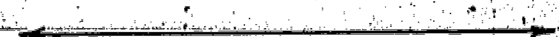
NUMBER LINE AND COORDINATES

Encourage students to draw number lines as well as to use those illustrated in the student's text. Call attention to the use of both vertical and horizontal number scales.

The thermometer scale was used to give a suggestion of a way of obtaining numbers associated with points on the left ray. It was used also to suggest that these numbers can be added. A method of adding is suggested by the method used in adding numbers on the thermometer scale.

After using the thermometer to motivate the introduction of negative numbers no further reference is made to it. Operations with negative numbers are developed by requiring that the commutative, associative, and distributive properties hold and by using some definitions--particularly additive inverses.

In Section 1-8 encourage students to be extremely careful in drawing the axes and labeling the scale on a pair of axes. Numbering should be below the X-axis and to the left of the Y-axis. The axes are lines and should be drawn with arrows as shown.



Always label the X-axis and the Y-axis as shown in the diagrams in the text.

Caution students to use a sharp pencil so they may have space to plot points and label them properly.

Answers to Exercises 1-1a

1. A: -4° D: 14°
   B: -2° E: -8°
   C: 10° F: 18°
   O: 0° G: -8°

2. No. 0° requires no symbol

3. (a) -6° (d) 0°
   (b) 3° (e) 10°
   (c) 8°

4. 20°; -10°; 0°; -10°; 10°

5.

Missing numbers in first column (in order)	In second column	In third column
0°	10°	20°
-30°	20°	0°
		70°
		60°

6.

Missing numbers in first column (in order)	In second column	In third column
-30°	40°	-40°
0°	10°	-30°
-10°		-60°
		-30°

Answers to Exercises 1-2b

1. (-1) is that number which added to 1 gives 0. The point associated with (-1) is located 1 unit to the left of point 0. Answers to other parts are similar.

2. Missing numbers: (a) 0, (b) 4, (c) 6, (d) 0, (e) $\frac{1}{2}$, (f) (-1/3), (g) 0, (h) 0

3. Additive inverses: (-7), 9, (-11), 12, 8, (-15), 20, 0, 2/3, (-4/9), (7/8), (-30/31)

Exercises 1-3-a Answers

1.		2.		3.	
(a)	4	(a)	3	(a)	(-5)
(b)	3	(b)	3	(b)	(-3)
(c)	5	(c)	8	(c)	(-6)
(d)	1	(d)	8	(d)	(-5)
(e)	2	(e)	8	(e)	(-10)
(f)	20	(f)	2	(f)	(-11)
(g)	10	(g)	9		
(h)	3	(h)	14		

Exercises 1-3-b Answers

1.		2.		3.	
(a)	(-5)	(a)	(-3)	(a)	2
(b)	(-3)	(b)	(-5)	(b)	(-2)
(c)	(-5)	(c)	(-11)	(c)	1
(d)	(-8)	(d)	(-10)	(d)	(-1)
		(e)	(-20)	(e)	8
		(f)	(-10)	(f)	(-5)
		(g)	(-20)	(g)	(-7)
		(h)	(-25)	(h)	6
				(i)	3
				(j)	(-5)

Exercises 1-3-c Answers

1. (a) (-9)
   (b) (-10)
   (c) (-64)
   (d) (-30)
   (e) (-50)
   (f) (-45)
   (g) (-37)
   (h) (-40)
   (i) (-29)
   (j) (-60)

Exercises 1-4-a Answers

1. (a) (-90)
   (b) (-72)
   (c) (-56)
   (d) (-2)
   (e) (-14)
   (f) (-169)
   (g) (-2646)
   (h) (-675)
   (i) (-18)
   (j) (-512)
   (k) 24
   (l) (-30)
   (m) 200
   (n) (-40)
   (o) 6

Exercises 1-4-b Answers

1. (a) 60
   (b) 12
   (c) 15
   (d) 44
   (e) 23
   (f) 300
   (g) 40
   (h) (-40)
   (i) (-40)
   (j) 42
   (k) 60
   (l) 110
   (m) (-192)
   (n) 192
   (o) 135
   (p) (-75)
   (q) 0
   (r) 16
   (s) 1000
   (t) (-60)
   (u) 60
   (v) 60
   (w) (-6)
   (x) 16
   (y) (-27)
   (z) 8

2. (a) $(-2)\cdot 6 = -12$
   (b) $5\cdot(-3) = -15$
   (c) $(-10)\cdot(-10) = 100$
   (d) $(-5)\cdot(-4) = 20$
   (e) $(-5)\cdot(4) = 20$
   (f) $11\cdot(-10) = 110$
   (g) $(-1)\cdot(-1) = 1$
   (h) $(-7)\cdot(0) = 0$
   (i) $1\cdot(-1) = -1$
   (j) $6 \cdot (-6) = -36$
   (k) $(-9)\cdot(9) = 81$
   (l) $5\cdot(-6) = -30$
   (m) $(9)\cdot(-10) = -90$
   (n) $(-2)\cdot(-50) = 100$
   (o) $(-6)\cdot(10) = -60$
   (p) $(-\frac{1}{2})\cdot(2) = -1$

Exercises 1-5 Answers

1.	(a)	2	(j)	21
	(b)	(-5)	(k)	-22
	(c)	(-5)	(l)	13
	(d)	5	(m)	-4
	(e)	(-5)	(n)	-25
	(f)	5	(o)	0
	(g)	5	(p)	13
	(h)	3	(q)	-12
	(i)	(-36)	(r)	-6

Exercises 1-6-a Answers

1.	(a)	(-2)	(e)	(-2)	(i)	(-4)	(m)	5
	(b)	(-3)	(f)	(-2)	(j)	(-2)	(n)	9
	(c)	(-2)	(g)	(-2)	(k)	(-1)	(o)	4
	(d)	(-4)	(h)	(-4)	(l)	(-4)	(p)	4

Answers to Exercises 1-6-b

1.	(a)	8	(e)	14	(i)	40
	(b)	13	(f)	12	(j)	25
	(c)	19	(g)	16	(k)	50
	(d)	20	(h)	12	(l)	80

Answers to Exercises 1-6-c

1.	(a)	(-10)	(e)	(-15)	(i)	(-6)
	(b)	(-11)	(f)	(-17)	(j)	(-8)
	(c)	(-15)	(g)	(-23)	(k)	(-14)
	(d)	(-16)	(h)	(-25)	(l)	(-12)

Answers to Exercises 1-6-d

1. (a) (-2) (e) 2 (i) 4
   (b) (-1) (f) 2 (j) (-3)
   (c) (-1) (g) (-2) (k) (-2)
   (d) (-3) (h) (-4) (l) 2

Answers to Exercises 1-6-e

1. (a) (-7) (f) (-6) (k) (-7)
   (b) (-2) (g) (-11) (l) 11
   (c) 4 (h) 12 (m) (-7)
   (d) 10 (i) 2 (n) 13
   (e) (-10) (j) 12 (o) 11

Answers to Exercise 1-7

2. $1\frac{1}{2}$ in., $4\frac{1}{2}$ in., 4 in., 1 in.
3. 1/3 mi., 3 mi.

Answers to Exercises 1-8-a

2. zero
3. zero
4. (0,0)
5. Opposite sides of the figure are parallel because they are segments of lines parallel to the x-and y-axes in a rectangular system of coordinates. Lines parallel to the y-axis are perpendicular to the x-axis by definition of a rectangular system of coordinates.

8

6. rectangle

7. (7,0)

8. yes, (0,0), 5 units.

Answers to Exercises 1-8-b

1. (3,5) -- I
   (1,-4) -- IV
   (-4,4) -- II
   (-3,-1)--III
   (8,6) -- I
   (7,-1) -- IV
   (-3,5) --III

3. isosceles triangle

5. A (3,5), B (-2,4), C (8,-2), D (9,9),
   E (-5,-5), F (-5,7), G (0,-4), H (-2,-9),
   J (8,3), K (-8,0), L (-7,-10), L (-7,-10), M (6,-8),
   N (7,11), P (-10,2), R (12,-5)

6. Quadrant I
   Quadrant III
   Quadrant IV
   Quadrant II

Unit II

E Q U A T I O N S

Outline of the Chapter

	Suggested time
Section 1. Orientation.	1 lesson
A. What solving equations means.	
B. Why solving equations is important.	
Section 2. Meaning of algebraic expressions.	2 lessons
A. Representation by computer diagrams	
1. Variables as inputs and outputs.	
2. Evaluation of algebraic expressions.	
B. Concepts of variable and constant.	
1. Domain of a variable.	
Section 3. Number sentences.	2 lessons
A. Concepts of equation and inequality.	
B. Number phrases and sentences.	
1. Solutions of number sentences in one variable.	
2. Representation on the number line.	
Section 4. Properties of equality relations.	1-2 lessons
A. Basic properties.	
1. Symmetry, reflexivity, and transitivity.	
2. Operations on equal numbers.	
B. Application to solution of equations.	
1. Logic of equation solving.	
Section 5. Number sentences in two variables.	2-3 lessons
A. Meaning of solutions.	

B. Geometrical representation. Plotting graphs.

Section 6. Solving for one variable in terms of the other.

A. Inverse operations. 1 lesson

Introduction

The teaching of this chapter will require careful planning of class work and assignments. Much of the text is written so as to lead the pupils to discover relationships for themselves. The teacher will have to generate many ideas in the class by questioning. Many of the exercises are designed to suggest ideas to the students. The teacher should note the remarks below on the exercises and be prepared for the class discussion. Since there are few routine problems in these exercises, no one pupil should be assigned too many at a time. It is better if a youngster does thoughtfully fewer problems than a larger number quite mechanically.

Tell the students that they cannot read mathematics like a novel. They should have pencil and paper handy while reading the text, and should work out the answers to the questions in each paragraph before going on to the next. They are not supposed to absorb these ideas like sponges. They must take an active part in the learning process. The sequence of questions in the text may suggest to the teacher how to develop with the pupils the concepts which are being developed.

At this time we are trying to convey the meanings of concepts related to solving equations, and to give the youngsters a technical vocabulary which they will need later. We are not trying to develop great skill or complete mastery right now. This will be left to the formal systematic instruction in algebra in the 9th grade. Do not dwell too long on this chapter, since the pupils will have ample practice with equations later.

1. Orientation

The objective of this section is to explain what solving equations means and why it is important. Do not spend too much time on this section. We are not aiming at skill or mastery at this stage.

You may have a little, but not much, practice at translating simple real-life situations into the form of equations. When you do this here, and later in the chapter, it is important to write explicitly on the blackboard such statements as

Let x = number of years Ann's age.

Emphasize to the class that when you formulate or solve a problem, that the thinking is presented in an orderly manner, and that anyone who comes into the room should be able to understand work on the board. Your chalk-board work should be a model for the way the children write up their homework problems.

It is a good idea to have as an exhibit in the classroom the equations brought in by the pupils in Exercise 2-1. You may divide the class into committees for the various subjects. The students may consult professional people in the community or at nearby colleges.

The references to the National Bureau of Standards are:

U. S. National Bureau of Standards. Applied Mathematics Series

No. 29 (1953). Simultaneous linear equations and the determination of eigenvalues. Edited by L. J. Paige and Olga Taussky.

No. 39 (1954). Contributions to the solution of systems of linear equations and the determination of eigenvalues. Edited by Olga Taussky.

U. S. Government Printing Office, Washington, D. C.

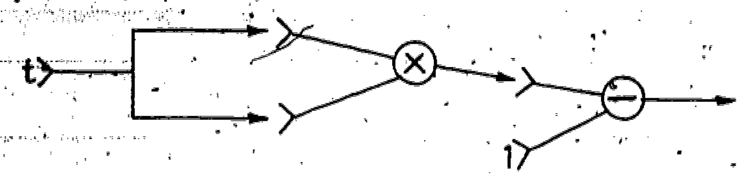
2. Meaning of Algebraic Expressions

The objective of this section is to teach the meaning of algebraic expressions. The notation is interpreted as instructions for the computation of the expression when values are given for the variables. This is made concrete by the device of computer diagrams. The variables are thought of as symbols for the inputs and the outputs. The evaluation of an expression is made concrete by the diagram.

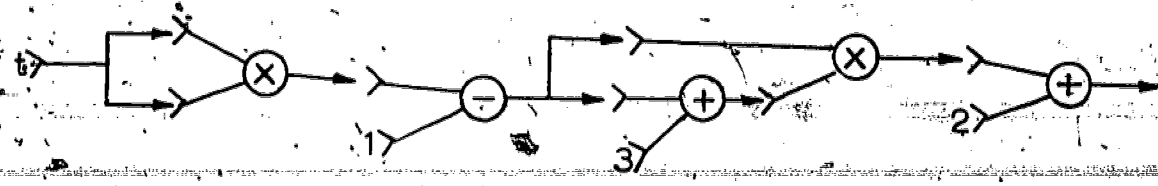
The grouping symbols are made concrete since they show that the results of certain computations are to be considered as single numbers to be fed in as inputs in further calculations. Thus in Problem 1(k) of Exercises 2-2a, one input to the multiplier is $(x+3)$. The output of this multiplier is $(x\cdot(x+3))$, which is one input of the adder. The output of the whole machine is

$$(x\cdot(x+3))+2$$

The operation of substitution can be made visual by means of these diagrams. To substitute $t^2 - 1$ for x in the above expression means to use as the input for x the output of the machine



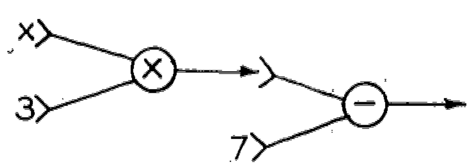
We can simplify simply hook up this machine to the other



and trace the flow of the computation.

The device also prepares for an understanding of the function concept. A function can be interpreted as a device which, for any input from a certain set, produces some definite output.

You may, in class, go through the process on P.3 with a different machine, say



and construct the table for $(x \cdot 3) - 7$. As in the text, you can first try particular numbers for x and tabulate the output. You may then give the class some outputs and ask them to find the inputs.

The questioning might go like this:

Suppose the output is 4.

What must the upper input of the subtractor be?

I have some money. I give away 7 dollars, and find that I have 4 dollars left. How much did I have before?

Now this number (11) must be the input x multiplied by 3.

If 3 times a number is 11, what is the number?

How can you undo multiplying by 3?

After the class understands what the machine does to actual numbers, you may then trace its operation on any input x, and obtain the algebraic expression $(x \cdot 3) - 7$. Now clinch the discussion by substituting values for x and verifying your table directly.

In Problem 2-2a-1, parts (a) and (b), (g) and (h), and (k) and (l) have the same outputs. The children should recognize this for the first two pairs immediately, and for the third after doing the next problem. A bright student may be able to apply the distributive and commutative laws to prove that

$$(x \cdot (x + 3)) + 2 = ((x \cdot x) + x \cdot 3) + 2 =$$
$$(x^2 + (3 \cdot x)) + 2$$

Do not worry if no one thinks of this at this time. A similar pair occurs in Problem 2-2b-1, parts (e) and (f).

Pupils should notice, similarly, that $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ by the distributive property.

In Problem 2-2b, problems 1(e) and (f), 3(a) and (b), 3(c) and (d) are related pairs. The whole set may be too much to assign to any one child for homework. It may be desirable to divide the class into subsets with different assignments. Each child should work on several of the related pairs so that he can have a chance to discover generalizations empirically. At this stage we do not care whether a pupil can prove such a relation as

$$(x + y) \cdot (x - y) = x^2 - y^2$$

or remember it.

Completed table for Section 2

x	y
0	6
1	9
2	12
5	21
8	30
-1	3
-2	0
4	18
13	45
-4	-6

Answers to Exercises 2-2a

1. (a) $x + 0$
   (b) $x \cdot 1$
   (c) $0 - x$
   (d) $\frac{1}{x}$
   (e) $x \cdot 0$
   (f) $(2 \cdot x) + 5$
   (g) $x + x$
   (h) $2 \cdot x$

(i) $x \cdot x$ (k) $(x \cdot (x + 3)) + 2$

(j) $\frac{x}{x}$ (l) $((x \cdot x) + (3 \cdot x)) + 2$

2.

	$x = 1$	$x = -2$	$x = 3$	$x = 0$
(a)	1	-2	3	0
(b)	1	-2	3	0
(c)	-1	2	-3	0
(d)	1	$-\frac{1}{2}$	$\frac{1}{3}$	will not accept zero
(e)	0	0	0	0
(f)	7	1	11	5
(g)	2	-4	6	0
(h)	2	-4	6	0
(i)	1	4	9	0
(j)	1	1	1	will not accept zero
(k)	6	0	20	2
(l)	6	0	20	2

No, all will not accept zero.

3. (a) 9

(b) 9

(c) -9

(d) $\frac{1}{9}$

(e) cannot do

(f) 2

(g) $\frac{9}{2}$ or $4\frac{1}{2}$

(h) $\frac{9}{2}$ or $4\frac{1}{2}$

(i) +3 or -3

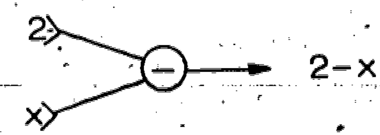
(j) cannot do

All of them cannot have the output 9.

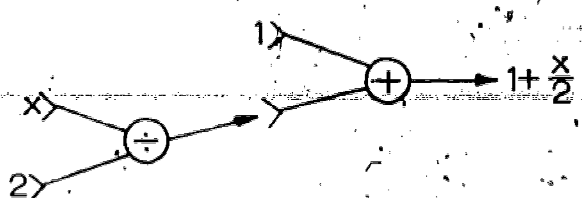
Answers to Exercises 2-2b

1.

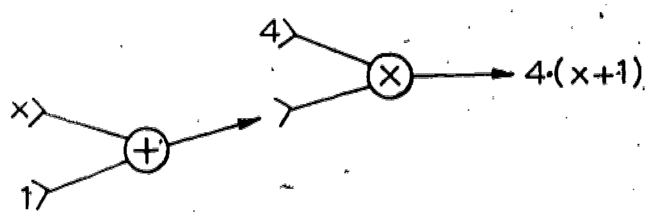
(a)



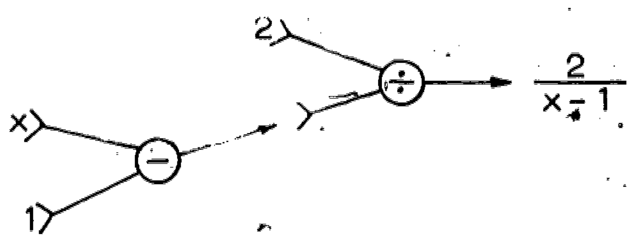
(b)



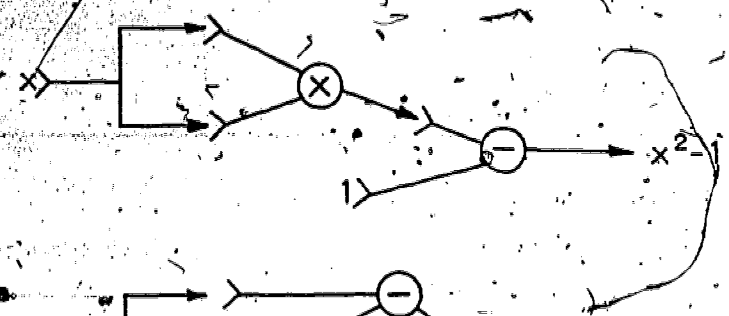
(c)



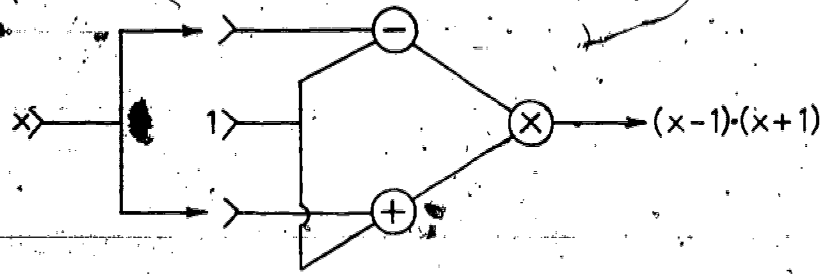
(d)



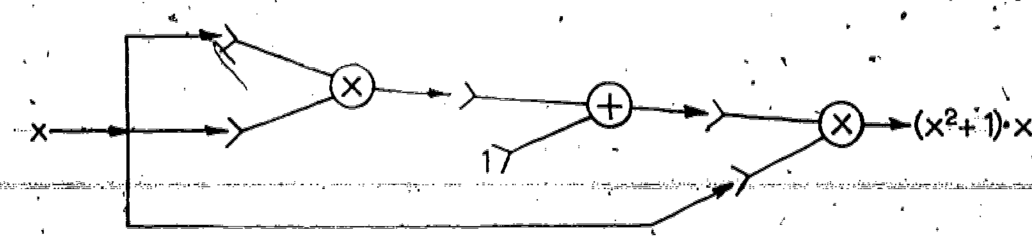
(e)



(f)



(g)



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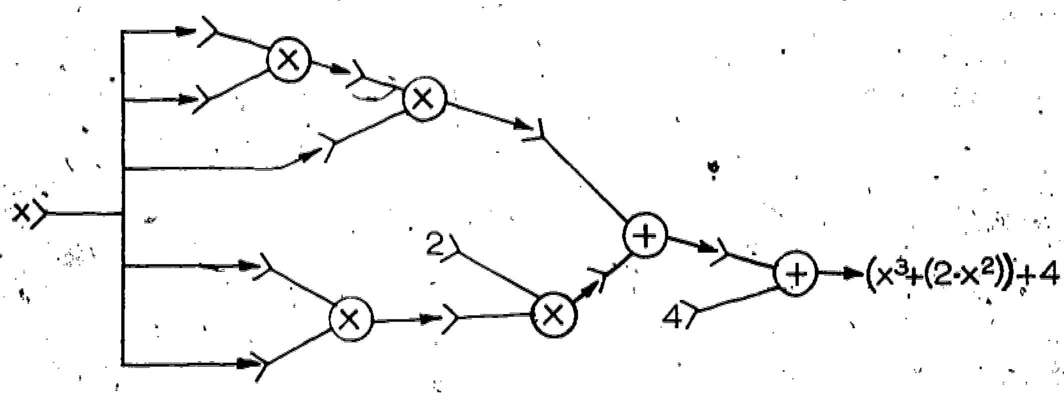


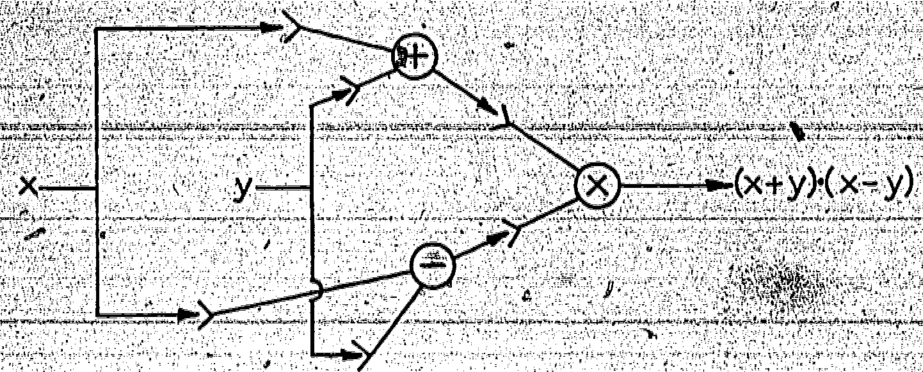
2.

	x = 0	x = 1	x = -2	x = 10
(a)	2	1	4	-8
(b)	1	$1\frac{1}{2}$	0	6
(c)	4	8	-4	44
(d)	-2	cannot do	$-\frac{2}{3}$	$\frac{2}{9}$

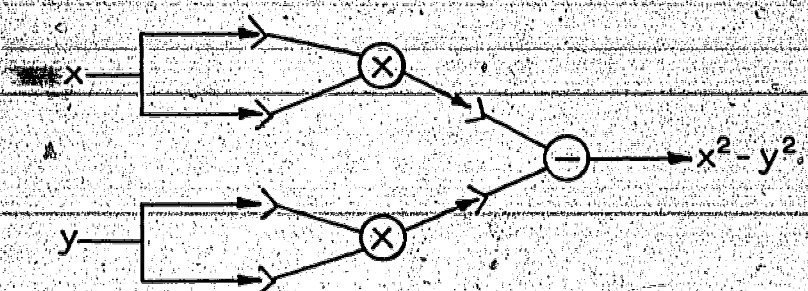
(e)	-1	0	3	99
(f)	-1	0	3	99
(g)	0	2	10	1010
(h)	4	7	4	1204

3.

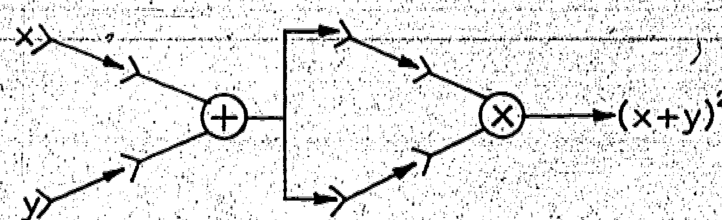
(a)



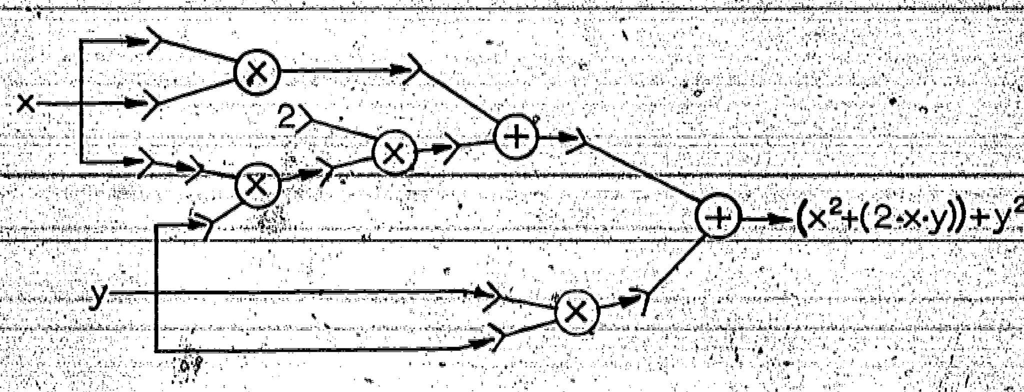
(b)



(c)



(d)



4.

x	1	2	10	30
y	2	1	1	5
(a)	-3	3	99	875
(b)	-3	3	99	875
(c)	9	9	121	1225
(d)	9	9	121	1225

5. (a) Positive numbers.

(b) Positive numbers up to limit of speedometer.

(c) Counting numbers up to limit of machine.

(d) Any number (absolute zero, -273° C. actually sets lower limit).

(e) Counting numbers.

3. Number Sentences

The object of this section is to build the student's mathematical vocabulary. He should understand the terms: number sentence, number phrase, open phrase, open sentence, equation, inequality, solution, solution set.

He should also be able to translate simple situations into mathematical language.

Notice that a sentence expresses a <u>proposition</u>, something proposed for consideration. It may be true or false.

You may add to the examples on P. 9 such sentences as

$$x = x + 3$$

or

$$0 \cdot x = 1$$

which are false no matter what number x is.

In Exercise 2-3a-1 note that such "axioms" as

"If equals are subtracted from equals, the results are equal."

are not needed to solve these equations. Any reference to such "axioms" makes the work more complicated than necessary. By the very meaning of subtraction, $a + x = b$ means the same as $x = b - a$.

So, in particular, the equation

$$z + 3 = -7$$

is equivalent to

$$z = (-7) - 3 = -10,$$

and similarly, the equation

$$7 - s = 2$$

is equivalent to

$$s + 2 = 7,$$

which is equivalent to

$$s = 7 - 2 = 5.$$

The relation between multiplication and division should be treated in the same way.

The students are not now expected to solve systems of equations. A bright youngster may discover the solution of the brainbuster by trial and error.

Answers to Exercise 2-3a

1. (a) 3 (b) 10 (c) -10 (d) -4

(e) $-\frac{5}{2}$ (f) $-\frac{7}{2}$ (g) $-\frac{2}{5}$ (h) $\frac{4}{3}$

(i) 5 (j) 9 (k) -2 (l) -5

(m) 9 (n) 5 (o) -98 (p) -180

2. (a) $y = 1958 - 14$

(b) $M = 13 - 5$

(c) $H = \frac{63}{3}$

(d) $t = \frac{500}{80}$

(e) $m = 2\frac{1}{2} - \frac{3}{4}$

(f) $90 \cdot x + 100(100 - x) = 9400$ (x is number of tons sent to Buffalo).

3. Seventy-five cows, 25 chickens. (The solution of this problem involves the solution of the equation $4 \cdot \text{cows} + 2 \cdot (100 - \text{Cows}) = 350$).

Answers to Exercise 2-3b

1. (a) {5}

(b) {4}

(c) {3}

(d) {20}

(e) {+3, -3}

(f) {0}

(g) cannot do

(h) {3}

(i) all numbers

(j) no solution

(k) {0, 3}

(l) any number less than 4

(m) any number greater than 4 and less than 100

(n) any number greater than 0 and less than 1.

(o) {-3,-3,-1,0,1,2,3}

(p) {1,2,3}

(q) {2,3,4,...,n} where n is a counting number.

(r) {6}

4. Properties of the equality relation.

In this section we are studying <u>relations</u> and their properties. We speak of <u>sentences</u> or <u>equations</u>. The mathematical verbs, "=", ">", and "<" are symbols for relations. Other examples of relations are "is the father of", etc. as in Exercise 2-4-4. We often use Latin letters, such as R, as symbols for relations.

You may read "aRb" as "a has the relation R to b."

We usually deal with relations in a certain set D, which is sometimes called the <u>universe of discourse</u>. In Section 4, the set D is the set of all numbers.

In this section we discuss the equality relation in a naive intuitive manner for children. The teacher should have a more mature understanding of the following concepts:

R is <u>reflexive</u> if xRx for all x in D.

R is <u>symmetrical</u> if xRy implies that yRx for all x and y in D.

R is <u>transitive</u> if, for all x, y, and z in D, whenever xRy and yRz, then xRz.

The equality relation has all three of these properties. The "<" relation is transitive, but neither reflexive nor symmetrical. The "≤" relation has the first and third properties, but not the second. The relation "is married to" is symmetrical, but neither reflexive nor transitive. If aRb means that a and b differ by less than 2, where a and b are numbers, then R has the first two properties but not the third. If R is symmetric and transitive and, for each x in D, there is a y such that xRy, then R is reflexive. For if xRy, then yRx, and since xRy and yRx, then xRx.

A relation which has all three properties is called an <u>equivalence</u> relation. Equality is a equivalence relation. "Has the same parents as" is an equivalence relation among human beings.

Properties 4 and 5 show the relation between the equality relation and the arithmetic operations. The "$<$" has these properties, too:

> If $x, y,$ and z are numbers and $x < y$, then $x + z < y + z$.
>
> If $x, y,$ and z are numbers and $x < y$ and $z > 0$, then $x \cdot z < y \cdot z$. Notice that if $z < 0$, then the last inequality would be reversed.

We apply the properties of equations to the solution of equations. Notice carefully the logic of the process. The usual reasoning in solving an equation actually proves a statement of the form:

> If x satisfies this equation, then $x =$ this or $x =$ that, etc. The logical function of checking is to prove the converse:
>
> If $x =$ this or $x =$ that, etc., then x satisfies this equation. Thus the process of checking is an essential

step in solving equations. Teach the pupils that their solution is not complete without the checking.

Answers to Class Exercise 4

1. (a) 4
   (b) 4
   (c) 4
   (d) 5
   (e) 5
   (f) 4
   (g) 5
   (h) 4
   (i) 4
   (j) 5
   (k) 4
   (l) 5
   (m) 5
   (n) 5

2. (a) No. 5 with $\frac{1}{2}$
   (b) No. 5 with $\frac{1}{4}$
   (c) No. 5 with 3
   (d) No. 5 with -1
   (e) No. 4 with -1
   (f) No. 5 with $\frac{1}{2}$
   (g) No. 5 with N
   (h) No. 5 with 10
   (i) No. 4 with 3x

3. (a) $y = 9$
   (b) $-3 + (2 \cdot y) = 5$
   (c) $7 = (3 \cdot w) + 1$
   (d) $6 = (3 \cdot w)$
   (e) $2 = w$
   (f) $t - 3.4 = -2.6$
   (g) $x = 108$
   (h) $(2 \cdot x) = 7 + (3 \cdot y)$
   (i) $y + (2 \cdot x) = 7$
   (j) $(y \cdot x) = 2 - (3 \cdot x)$
   (k) $y = \frac{2}{x}$

Answers to Exercises 2-4

In problem 1 the teacher may want to ask for more intermediate steps with qualifying properties such as associativity and the inverse relation, see complete discussion of examples in text, Section 4.

1. (a) $(2 \cdot x) + 1 = 7$

$2 \cdot x = 6$ Property 4

$x = 3$ Property 5

(b) $y - 2 = 6$

$y = 8$ Property 4

(c) $\frac{t}{2} - 3 = -4$

$\frac{t}{2} = -1$ Property 4

$t = -2$ Property 5

(d) $(3 \cdot x) - 5 = (2 \cdot x) + 3$

$x - 5 = 3$ Property 4

$x = 8$ Property 4

2. (a) 2
(b) -8
(c) 1
(d) -4
(e) 2
(f) -2
(g) 8
(h) 8
(i) 4
(j) 4
(k) -4
(l) 2

3. (a) Properties 3,4,5 unless $c = 0$.
(b) None
(c) Property 3
(d) Properties 1,2,3
(e) Property 2

5. Number Sentences in Two Variables.

It is especially important that the students work through this section with pencil and paper. They should all have graph paper before they begin reading this part. You may pattern the class discussion on the questions in the text.

During the first lesson in this section you should work out a number of different types of problems in class. You may graph such number sentences as

$$2 \cdot x = y,$$

$$y = x^4,$$

x and y are non-negative integers and

$$x + (2 \cdot y) = 13,$$

$$x \geq 0,\ y \geq 0, \text{ and } x + y = 10.$$

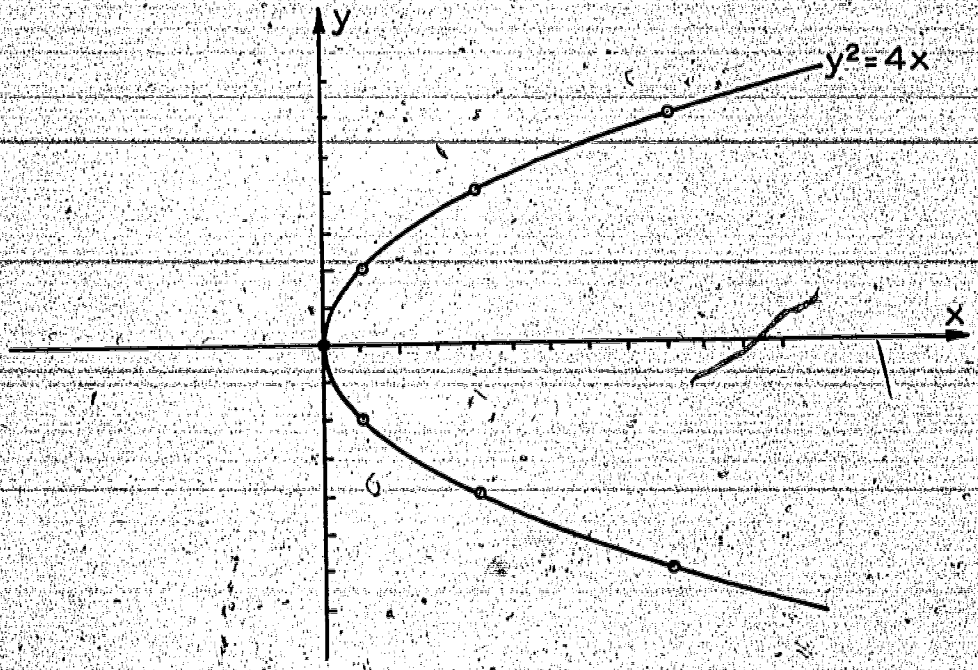
Answers to Questions in 5.

1st Table, Section 5,

x	y
0	1
1	2
2	3
-1	0
$-\frac{2}{3}$	$\frac{1}{3}$
$-\frac{16}{3}$	$-\frac{13}{3}$

2nd Table, Section 5,

x	y
0	0
4	4
4	-4
.9025	1.9
1.1025	2.1
9	6
-1	cannot do



In class discussion of $y^2 = 4x$, the students should notice that for each positive number x there are two values for y, but for any negative number x there are no values for y. You may find it worthwhile to make a magnified graph of the neighborhood of the origin, using $y = 0, \pm .1, \pm .2$, etc.

There are more problems in Exercises 2-5 than should be assigned to any one pupil. Divide the class into subcommittees with responsibility for different sets of problems. The graphs should be posted around the room so that all the children will see the great variety of graphs. Certain families should be exhibited on the same paper so that the youngsters will have a chance to discover generalizations.

Examine these exercises before the class discussion in order to be prepared for the students' questions.

The brainbuster (Problem 8) is an excellent topic for research project for a science fair or a junior academy of science. The students may investigate other equations of the form $(a \cdot x) + (b \cdot y) = n$, where a, b, and n are positive integers. Some may work on the solution in non-negative integers of equations in more than two unknowns, such as

$$x + (2 \cdot y) + (3 \cdot z) = n, \quad x + (2 \cdot y) + (3 \cdot z) + (4 \cdot t) = n, \text{ etc.}$$

The solutions of

$$(2 \cdot x) + (3 \cdot y) = n$$

in non-negative integers are given by

$$x = (3 \cdot k) - n, \quad y = n - (2 \cdot k),$$

where k is any integer such that

$$\frac{n}{3} \leq k \leq \frac{n}{2}$$

The integer k can be computed from the equation

$$k = x + y.$$

The greatest value of k for a given n is the integral part of $\frac{n}{2}$. The smallest value of k for a given n is $\frac{n}{3}$, if n is divisible by 3, and otherwise 1 is more than the integral part of $\frac{n}{3}$. The number of solutions for a given n is 1 more than the difference between the largest and smallest values of k.

The solutions may be classified according as $y = 0$ or $y > 0$. There is a solution with $y = 0$ if and only if n is even. For any solution with $y > 0$ we must have $y = 1 + z$, where z is a non-negative integer. Then

$$n = (2 \cdot x) + (3 \cdot y) = (2 \cdot x) + (3 \cdot (1 + z)) = ((2 \cdot x) + (3 \cdot z)) + 3,$$

so that (x,z) is a solution of

$$(2 \cdot x) + (3 \cdot z) = n$$

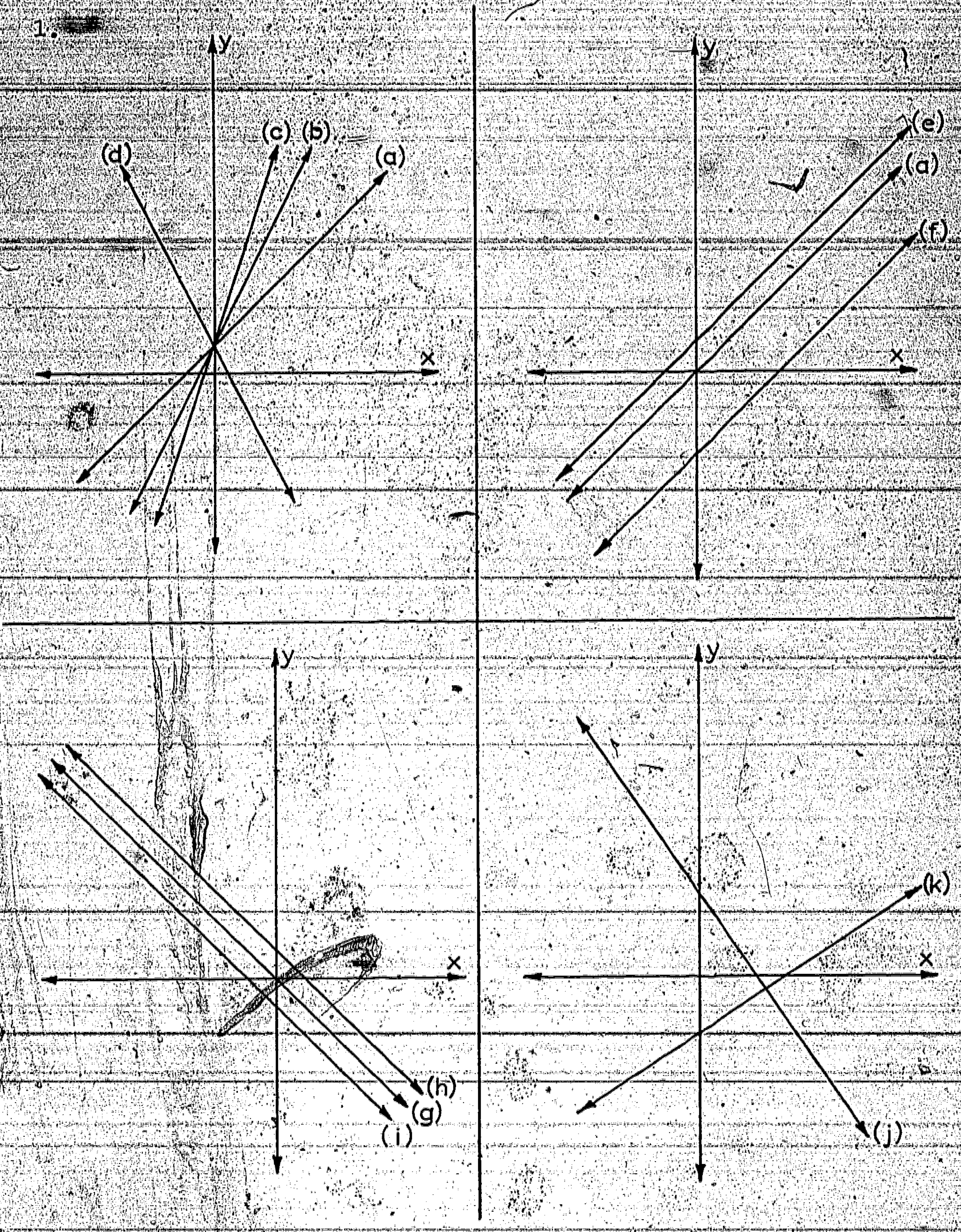
Hence the number of solutions for a given n with $y > 0$ is the same as the number of solutions for $n - 3$. If A_n = the number of solutions of $(2 \cdot x) + (3 \cdot y) = n$ in non-negative integers, then

$$A_n = A_{n-3} + \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

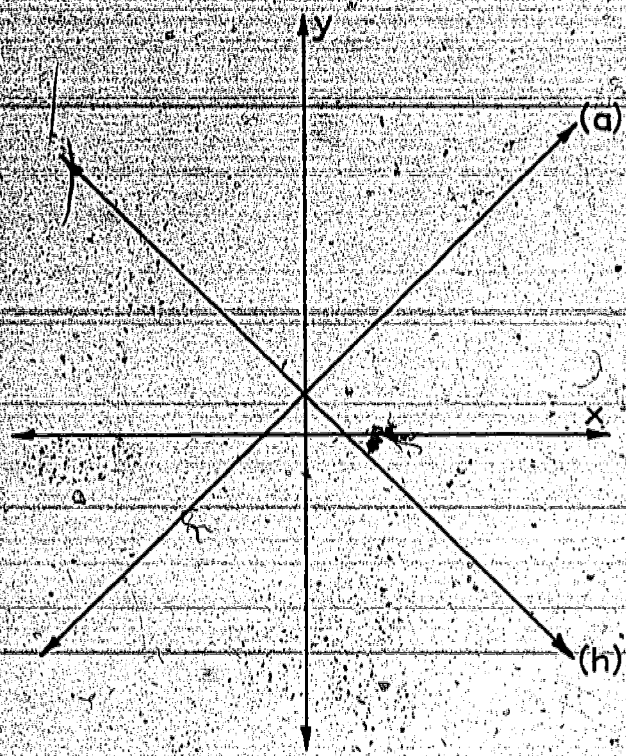
All of this is information to show how much there is for the students to discover in this problem. Tell them as little as possible except to explain the meanings of the terms and to suggest things to look for. The important point is for them to begin exploring and investigating for themselves.

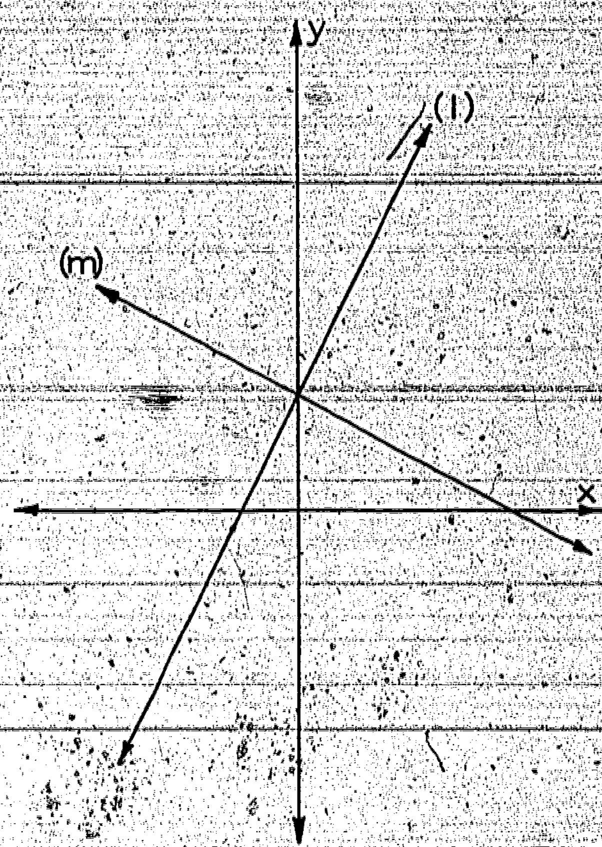
Answers to Exercises 2-5

1.

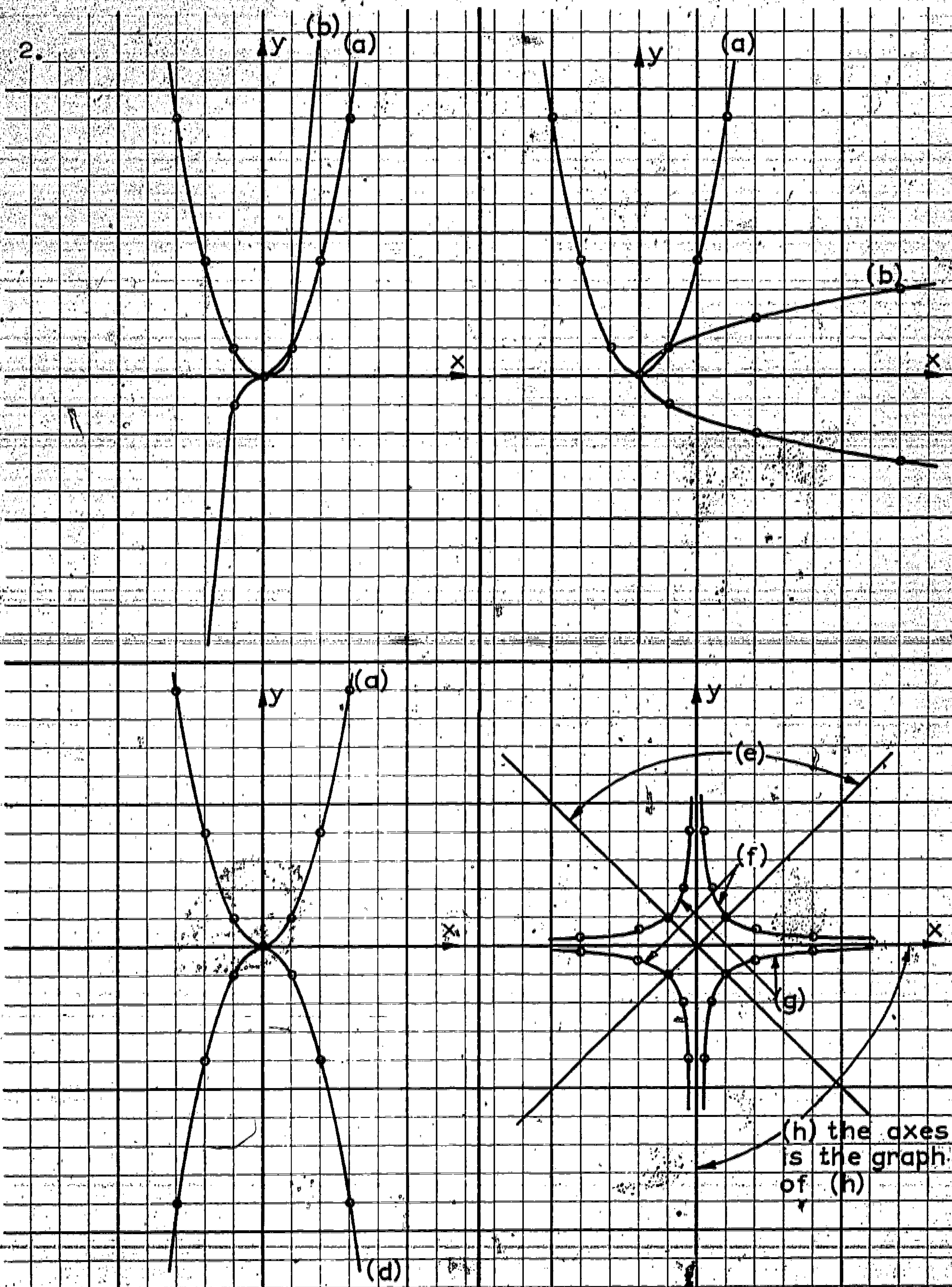


1 (cont'd)

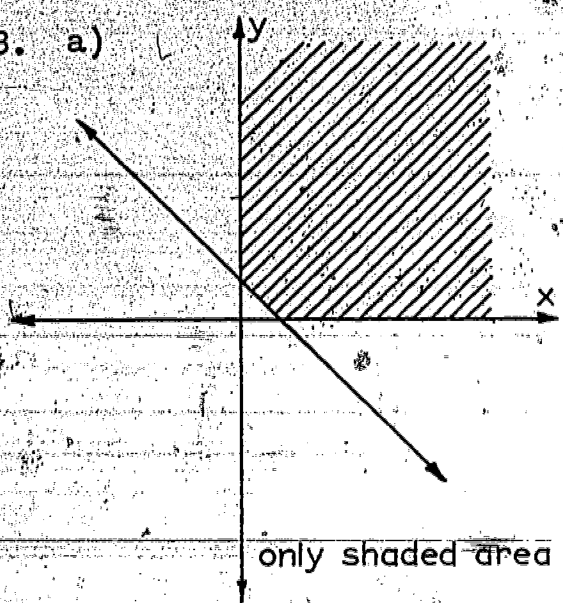




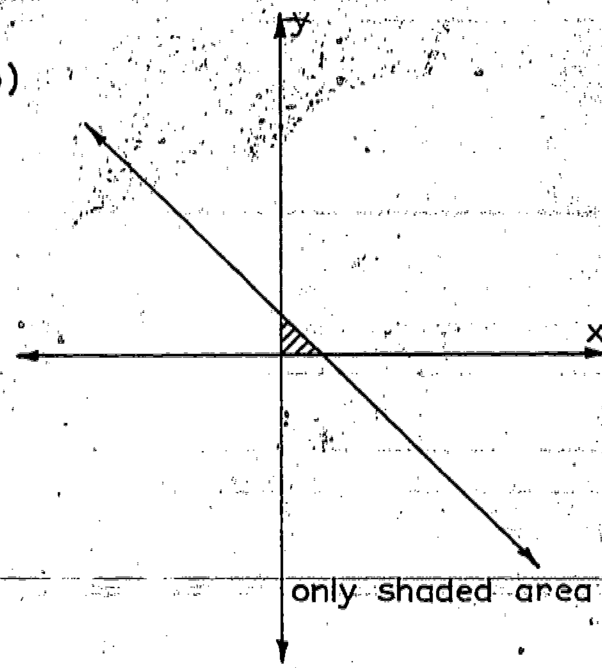
2\.



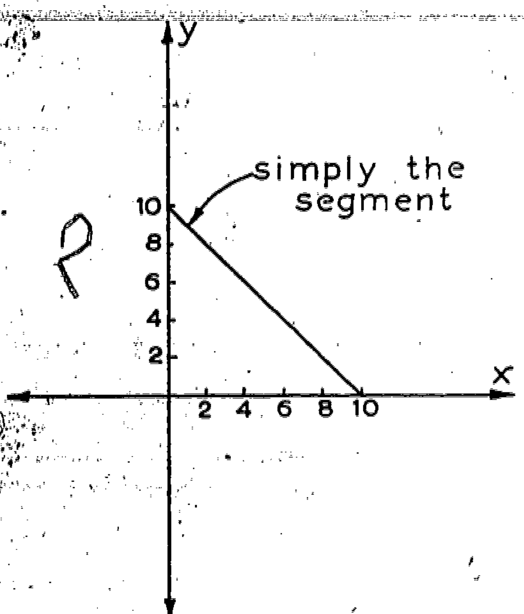
3. a)



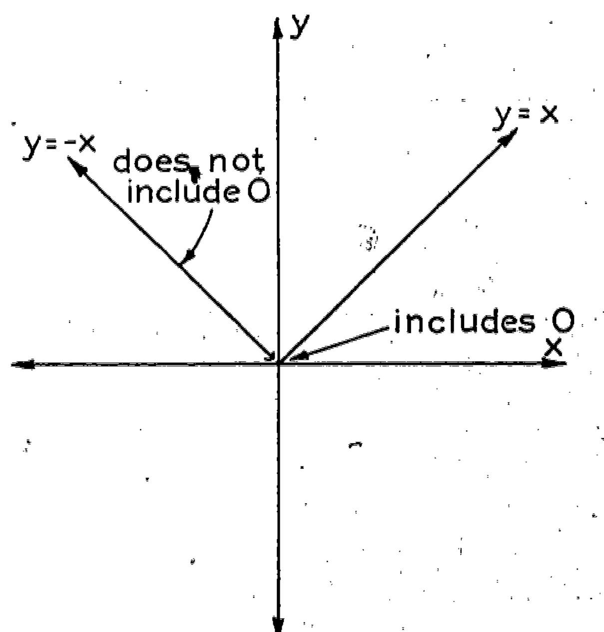
b)



c)



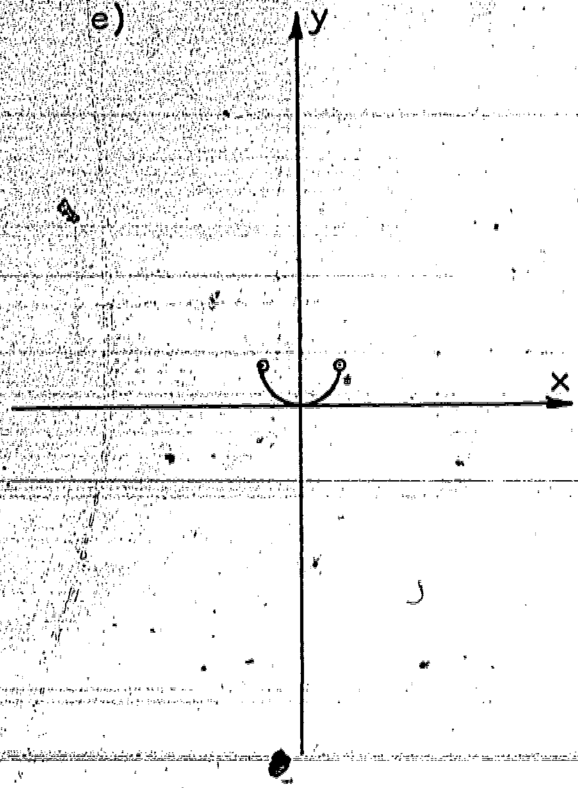
d)



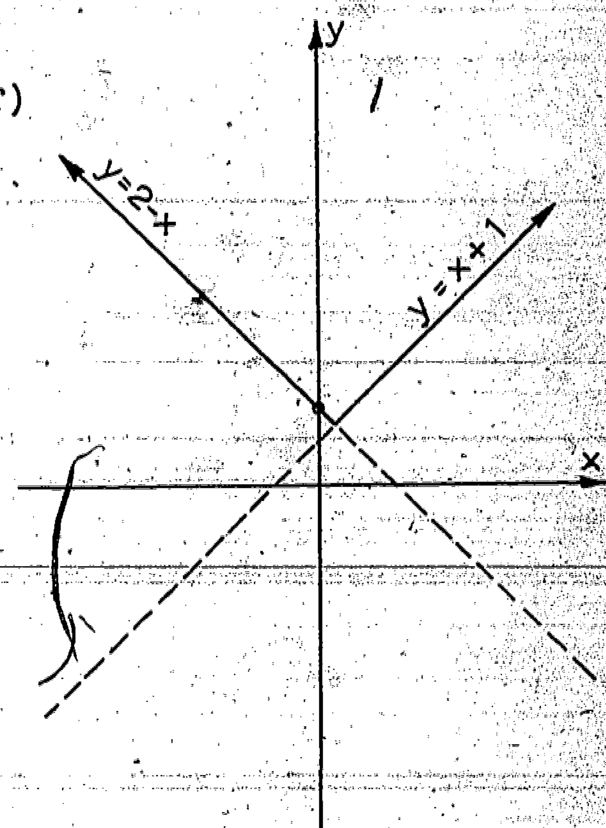
graph is the intersection of the two lines above which is the empty set

3) (cont'd)

e)

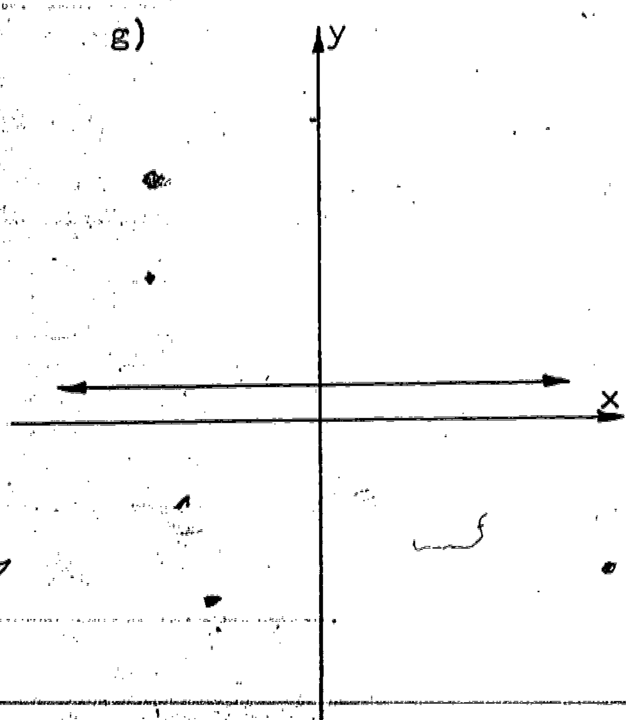


f)

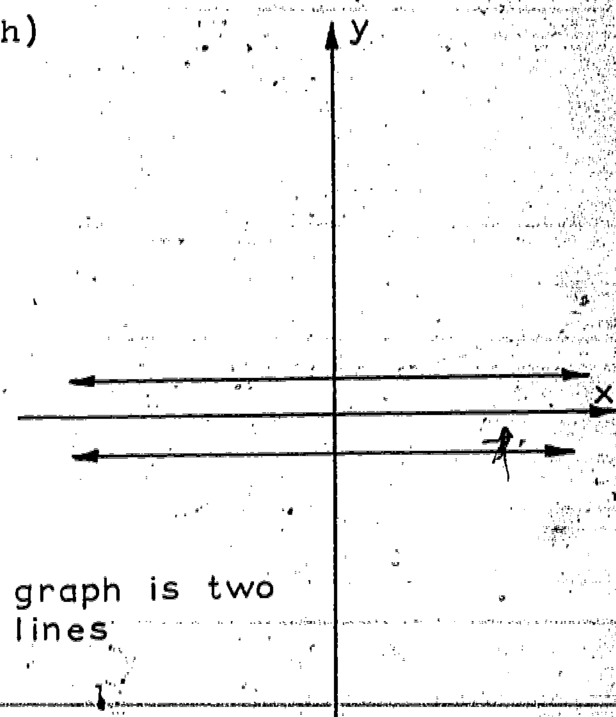


graph is solid line

g)



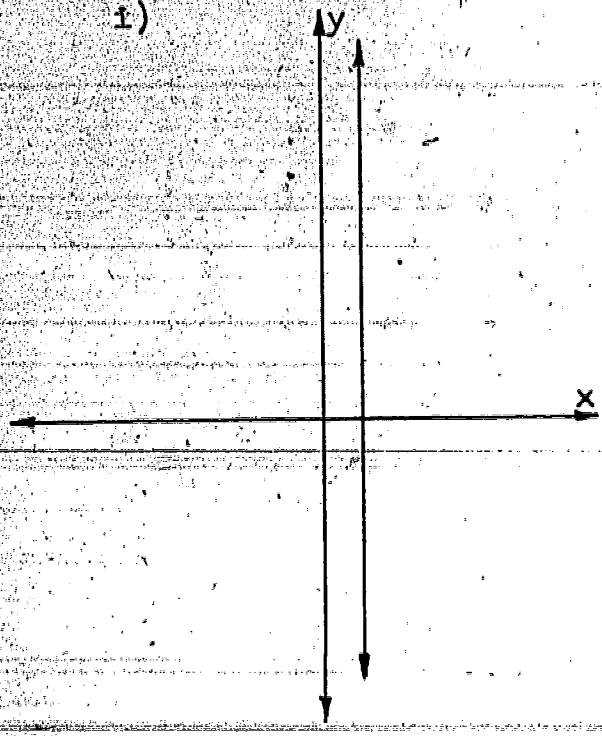
h)



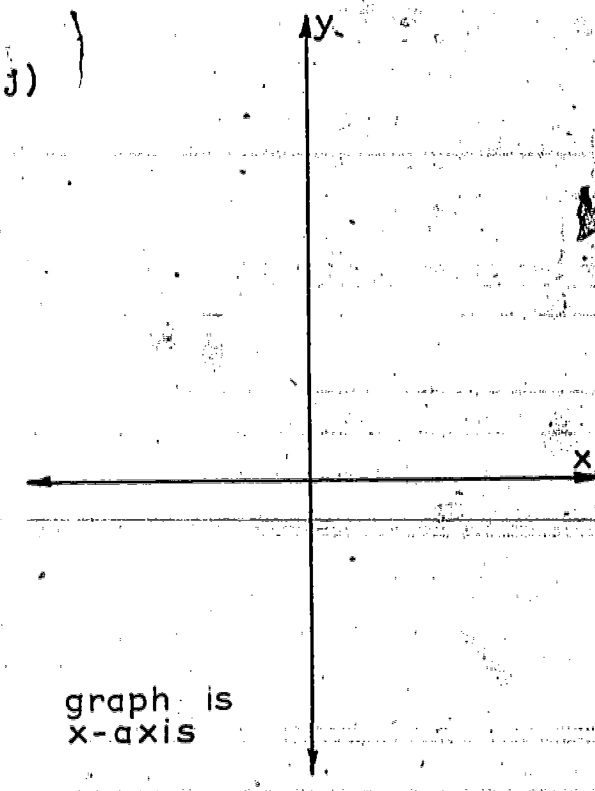
graph is two lines

3) (cont'd)

i)

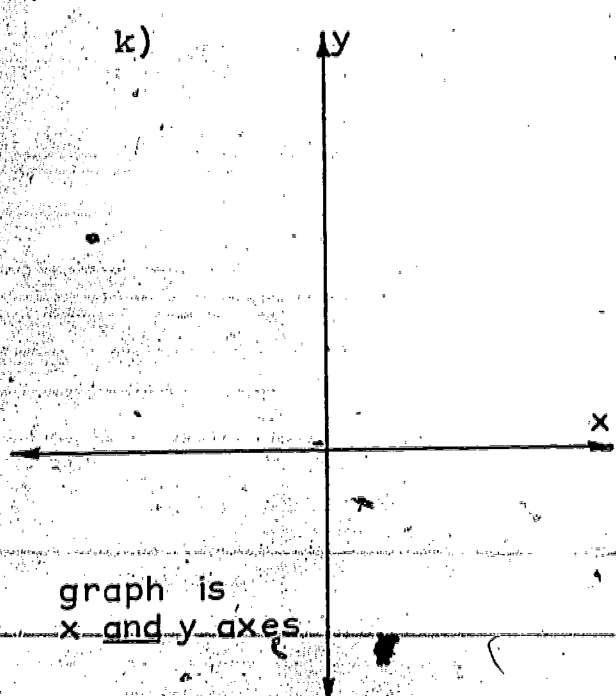


j)



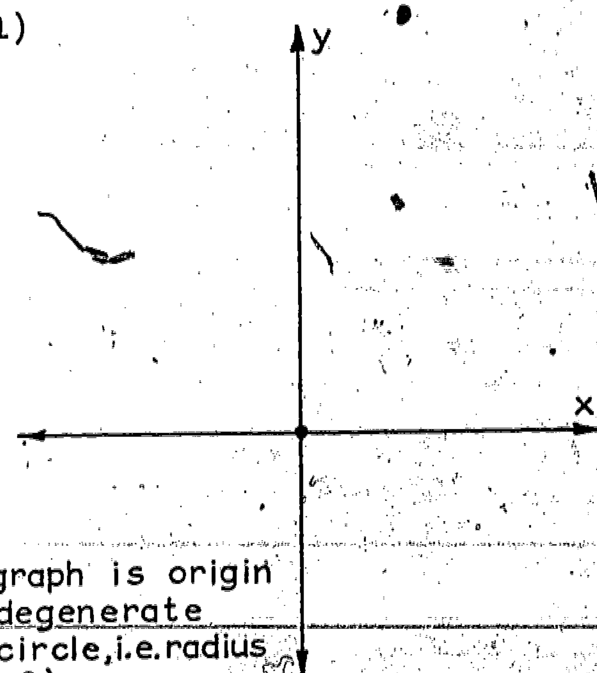
graph is
x-axis

k)



graph is
x and y axes

l)



graph is origin
(degenerate
circle, i.e. radius
=0)

4. (a) $\{(0, 1), (1, 0)\}$

(b) $\{(0, 2), (1, 1), (2, 0)\}$

(c) $\{(0, 20), (1, 19), \ldots (10, 10) \ldots (19, 1), (20, 0)\}$

(d) $\{(0, 0)\}$

(e) $\{(1, 0)\}$

(f) $\{(2, 0), (0, 1)\}$

(g) $\{(3, 0), (1, 1)\}$

(h) $\{(0, 2), (2, 1), (4, 0)\}$

~~(i) {(1, 12), (3, 11), (5, 10) .... (21, 2), (23, 1), (25, 0)}~~

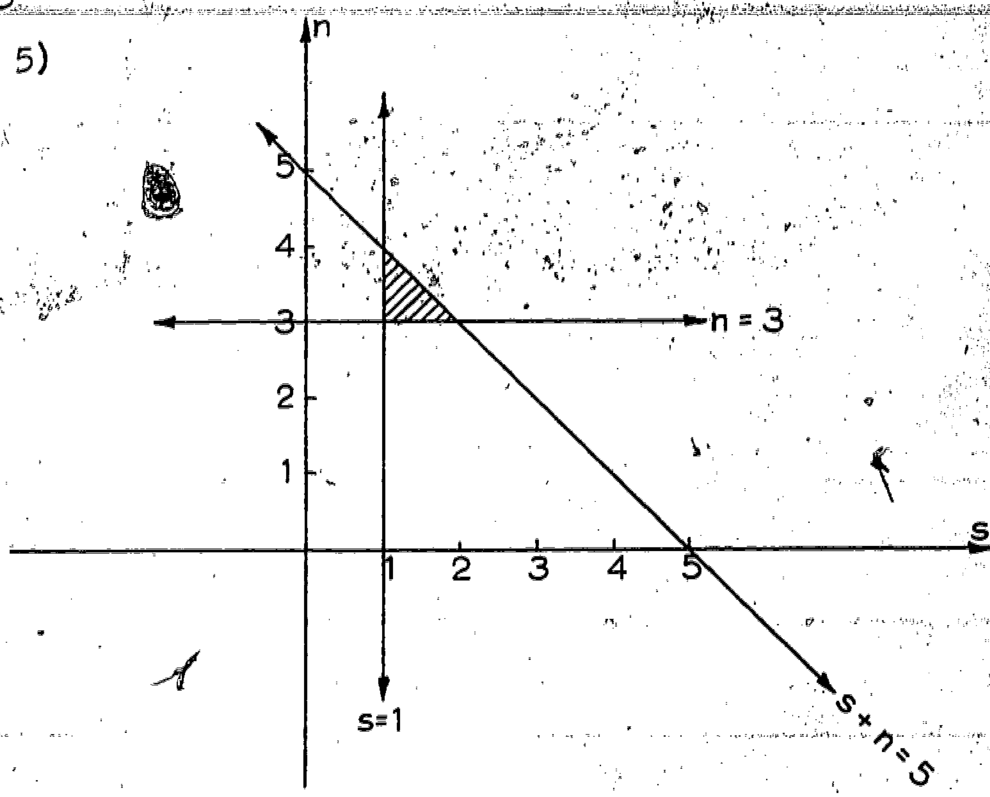
(j) $\{(0,5), (7, 0)\}$

(k) $\{(3, 3)\}$

(l) $\{(6, 1)\}$

Problem 5

(Section 5)



The number sentence which describes the new situation completely and is shown on the graph is:

$$S + N \leq 5 \text{ and } N \geq 3 \text{ and } S \geq 1.$$

6. Solving for One Variable in Terms of the Other

The purpose of this section is to give the pupils a taste of the solution of equations like

$$(2 \cdot x) - (3 \cdot y) = 7 \quad \text{(to be solved for } x)$$

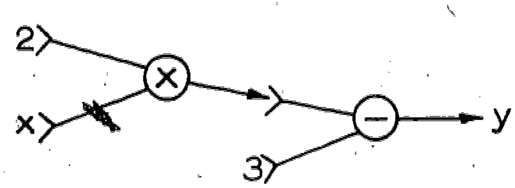
whereas up to now in all the examples the solution was always a particular number. We try to make the children see this problem as a generalization of equations like

$$(2 \cdot x) - (3 \cdot 5) = 7.$$

In discussing equations like

$$(2 \cdot x) - 3 = y$$

the students are developing the concept of an <u>inverse</u> operation. The machine

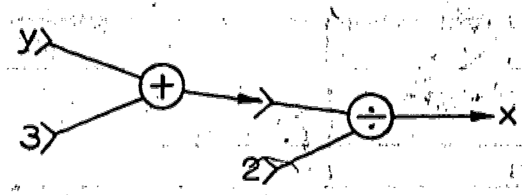


performs a certain operation on the input x to produce the output y. We must design a machine which operates on x and produces y.

We can do this by retracing the operations. If you know y, what must you do to obtain the upper input of the subtracter? This number $- 3$ is y; what is the number? Clearly, to obtain this number you must <u>undo</u> subtraction of 3. You must <u>add</u> 3.

Now if $2 \cdot x$ is this number, what must you do to obtain x? How can you undo multiplication by 2? Obviously, you must divide by 2.

This leads to the design



and the solution

$$x = \frac{y + 3}{2}.$$

A similar series of questions will lead the children to solve other problems of this type.

In the exercises there are problems like

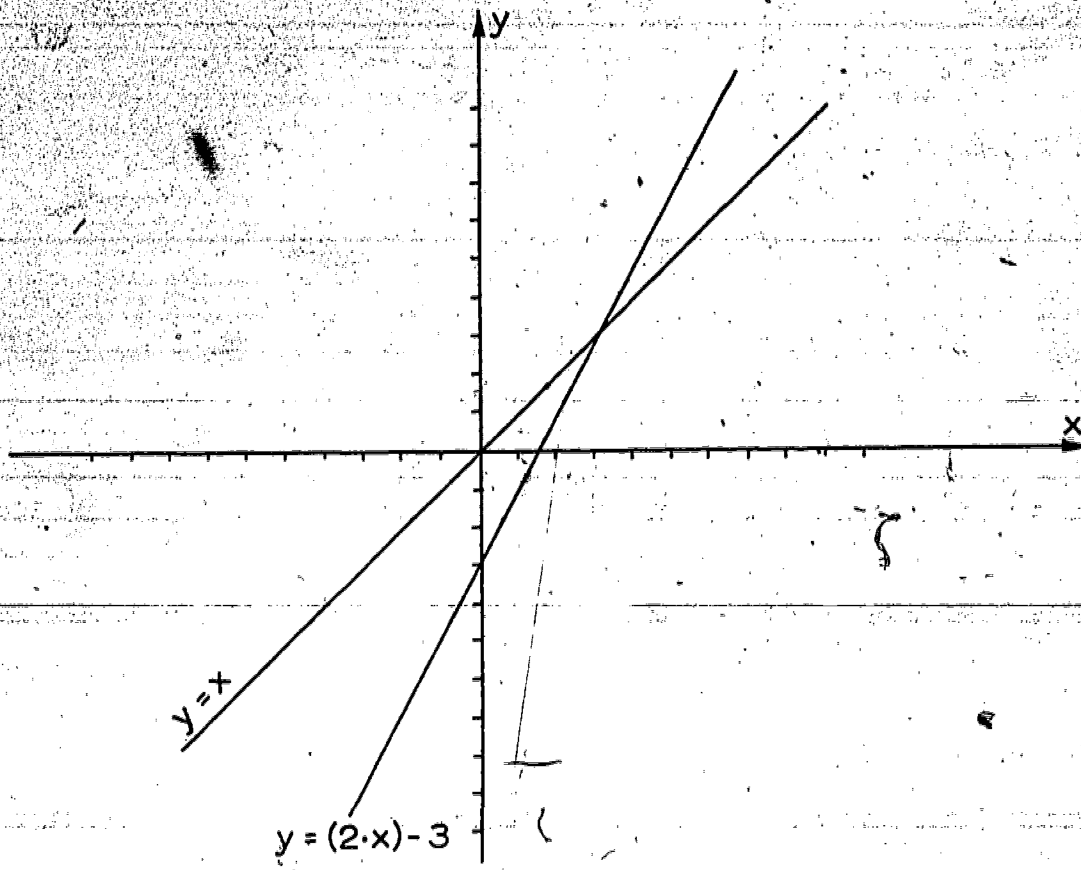
$$y = (2 \cdot x) - 3 \quad \text{and} \quad y = x.$$

You may present the problem like this: what number x is left unchanged by the operation $(2 \cdot x) - 3$? Graphically, we are looking for the intersection of the like

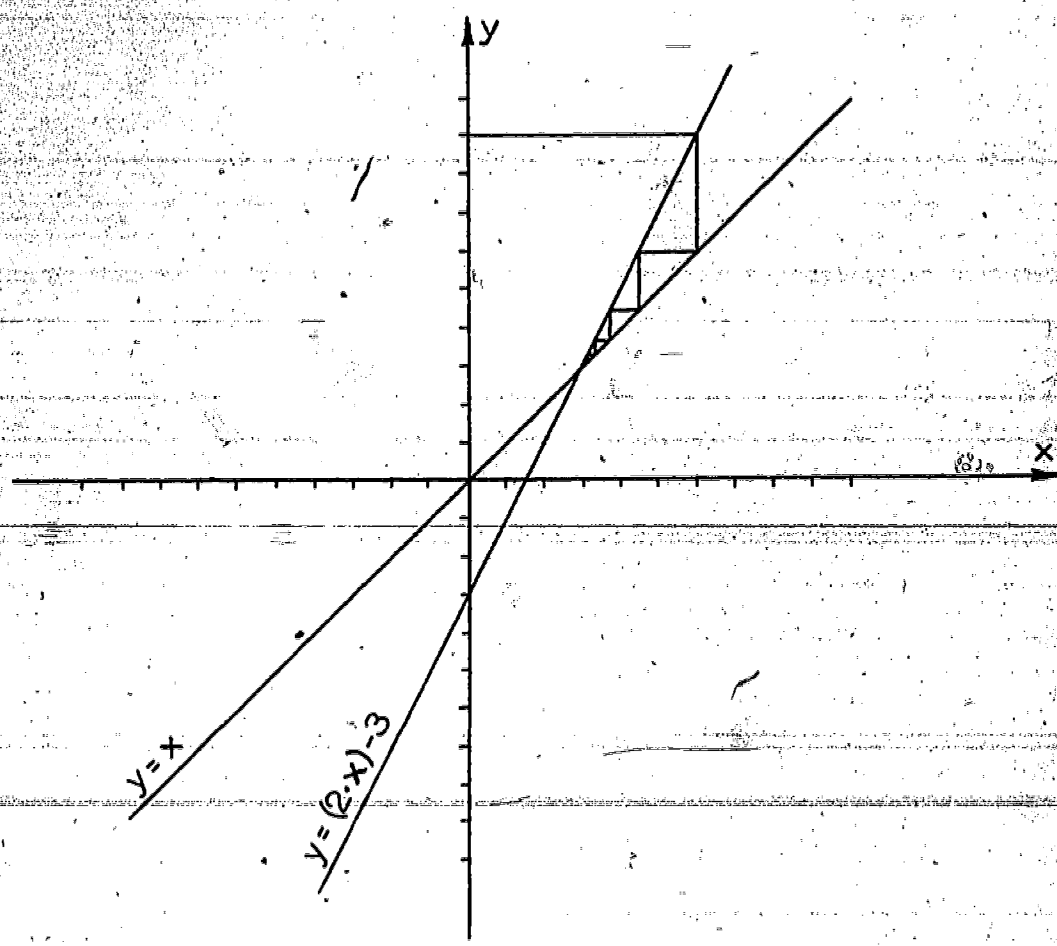
$$y = (2 \cdot x) - 3$$

with the line

$$y = x.$$



An amusing way to solve the problem is to start with a guess at y, and compute x. Then use this as your new guess for y, and repeat. As we continue this process we obtain better and better approximations to the exact intersection:



The brainbuster can be interpreted in a similar way. We can see that the equation

$$x = \frac{1}{2} \cdot \left(x + \frac{a}{x}\right)$$

is equivalent to

$$2 \cdot x = x + \frac{a}{x}$$

and to

$$x = \frac{a}{x}$$

or

$$x^2 = a.$$

Therefore the problem of computing $\sqrt{a}$ is equivalent to finding the number x which is left unchanged by the operation.

$$\frac{1}{2}\left(x + \frac{a}{x}\right)$$

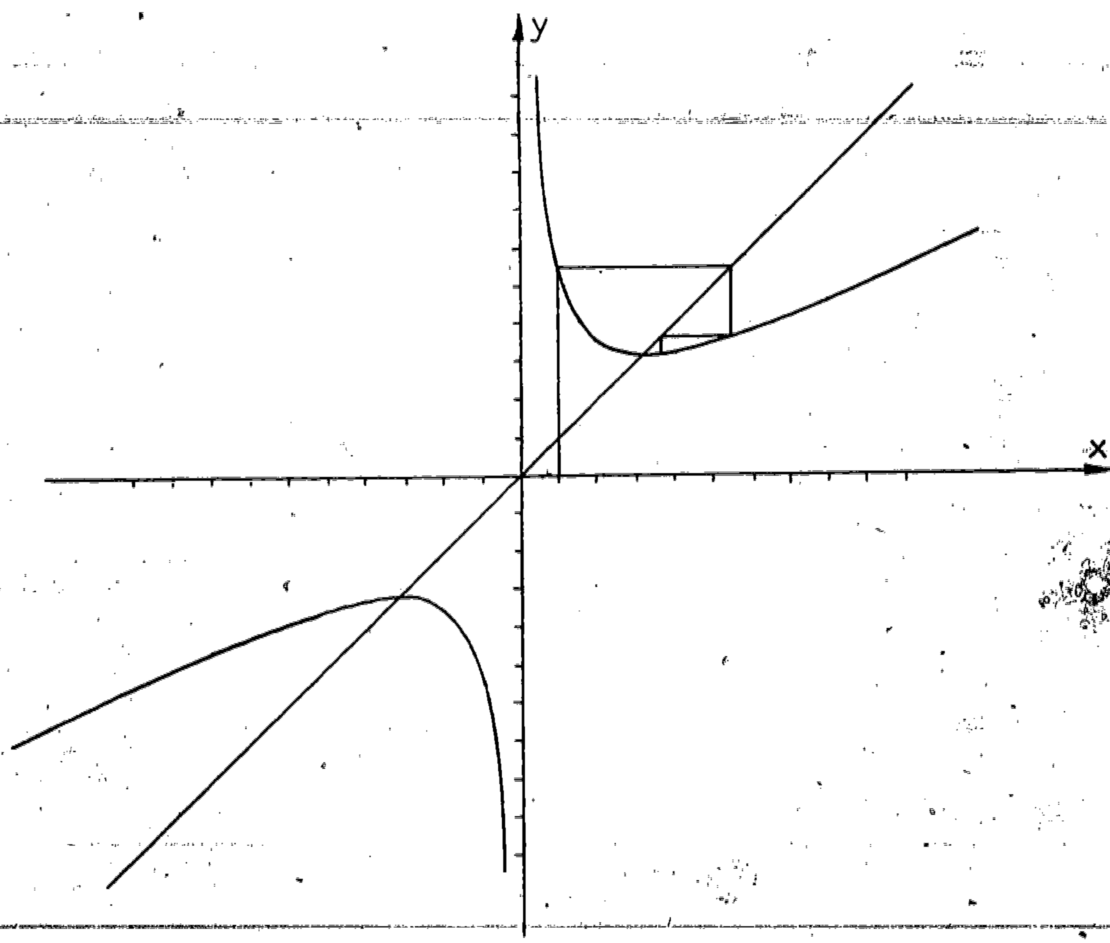
or to finding the intersection of the graph of

$$y = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

with the line

$$y = x$$

Here is the picture with $a = 10$. In the exercise we have taken $a = 4$, so that $\sqrt{a} = 2$.



We can begin with a guess for x, say $x = 3$, and compute y from the equation

$$y = \frac{1}{2}\left(x + \frac{10}{x}\right)$$

(We obtain $\frac{19}{6}$). We use this as our new guess for x, and repeat. We obtain a sequence of numbers which approach $\sqrt{3}$, as can be seen from the figure.

The above procedure, sometimes referred to as "iterative methods," is described further in an article entitled "Feed it Back" by Francis Scheid, The Mathematics Teacher, April 1959, pp. 226 - 229.

Answers to Exercises 2-6

1. (a) $\frac{y - 1}{2}$ (h) $\frac{y - 5}{2}$

   (b) $\frac{y + 3}{2}$ (i) $\frac{y - 7}{-2}$ or $\frac{7 - y}{2}$

   (c) $\frac{7 - y}{2}$ (j) $\frac{7 + y}{2}$

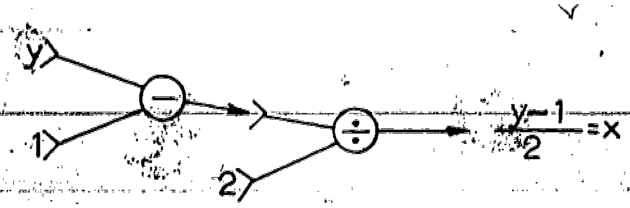
   (d) $\frac{7 - (3 \cdot y)}{2}$ (k) $\frac{7 + (3 \cdot y)}{2}$

   (e) $2y$ (l) $2y + 1$

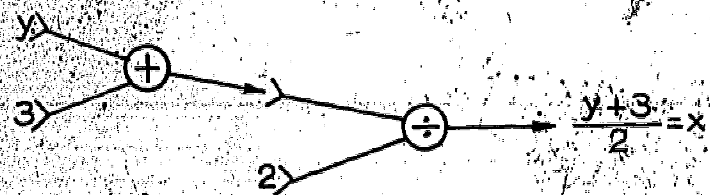
   (f) $\frac{2}{y}$ (m) $\frac{2}{y + 3}$

   (g) $\frac{2}{y}$ (n) $\frac{2}{y + 3}$

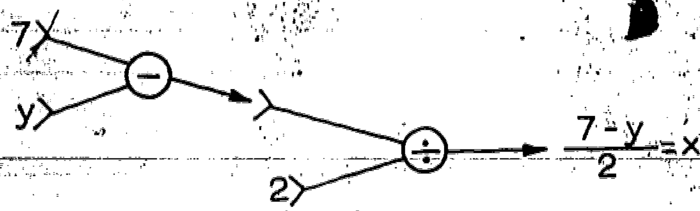
2. (a)



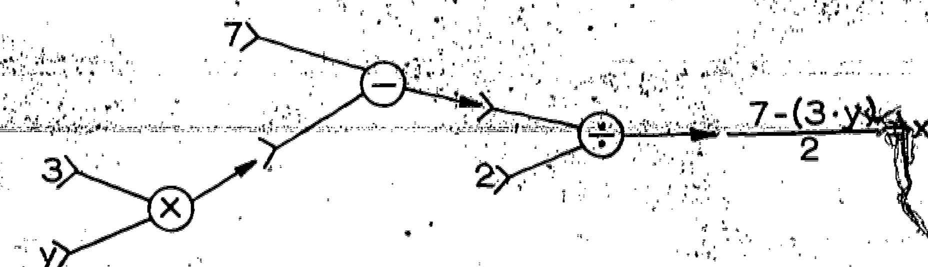
(b)


y
3
2
$\frac{y+3}{2} = x$

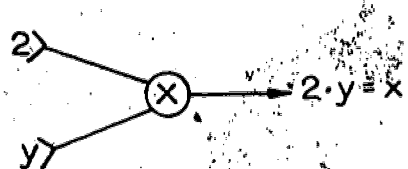
(c)


7
y
2
$\frac{7-y}{2} = x$

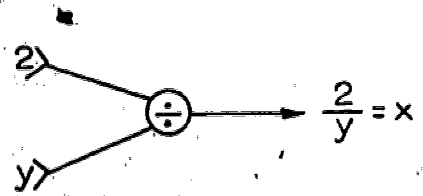
(d)


7
3
y
2
$\frac{7-(3\cdot y)}{2} = x$

(e)

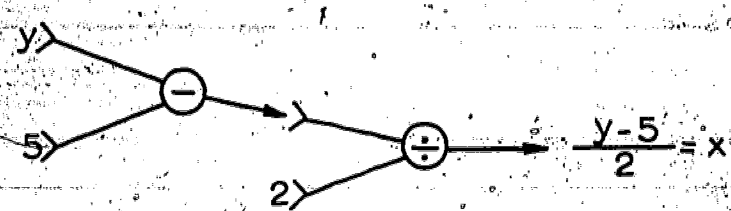

2
y
$2\cdot y = x$

(f)

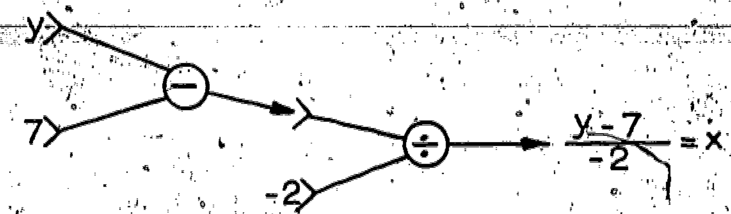

2
y
$\frac{2}{y} = x$

(g) Same as (f)

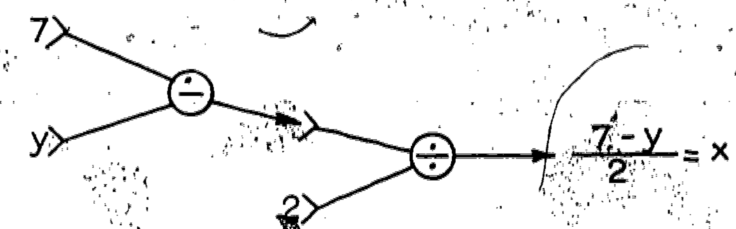
(h)



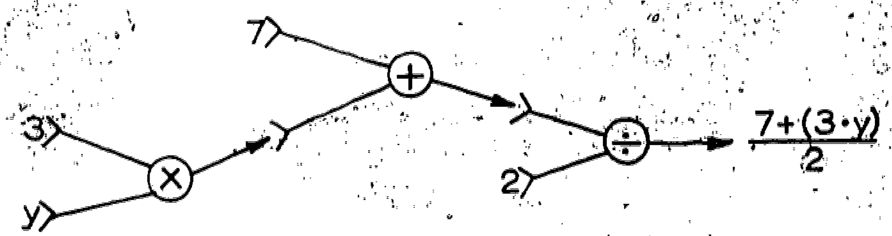
(i)



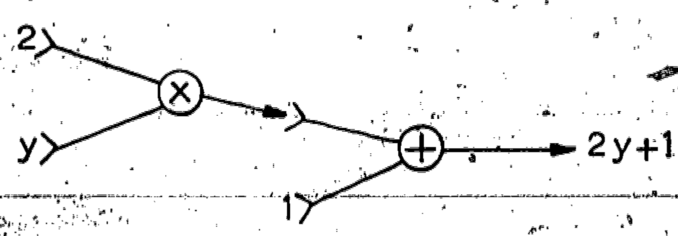
(j)



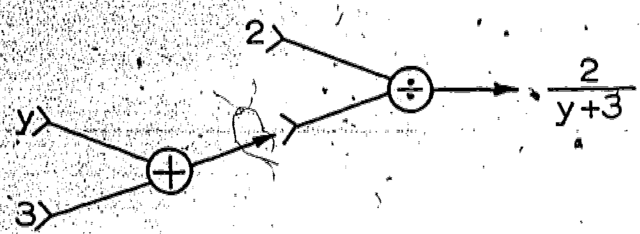
(k)



(l)



(m)



(n) Same as (m)

3. (a) -3

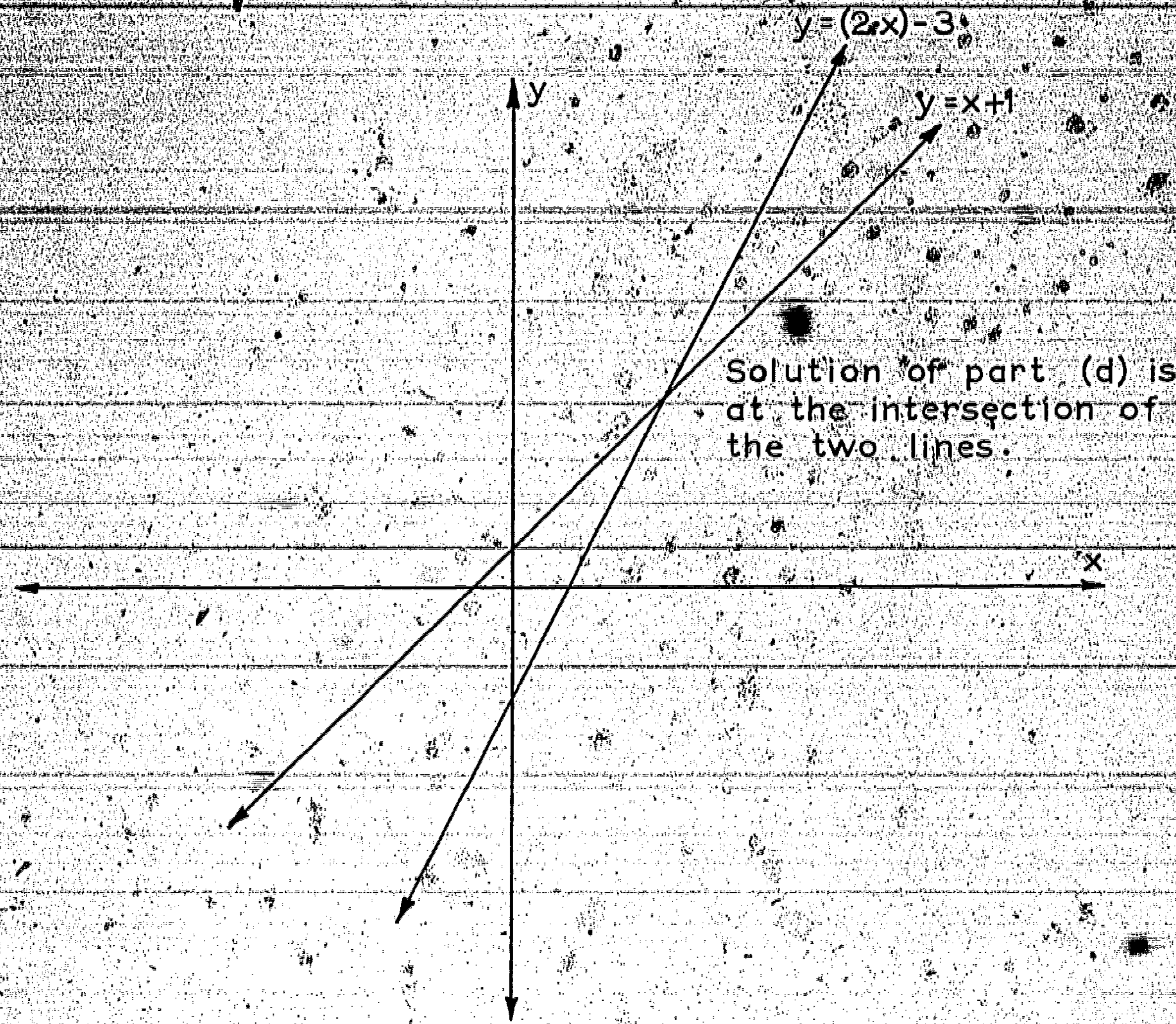
(b) $x = -3$, $y = -3$

(c) $x = 3$, $y = 3$

(d) $x = 4$, $y = 5$

3. (e)

Section 6)



4. Brainbuster: { -2, 2}.

Unit III

S C I E N T I F I C   N O T A T I O N ,
A P P L I C A T I O N S   O F   P E R C E N T

For this unit it is assumed that the student has had some acquaintance with the names of numbers, the decimal notation, and finding products involving decimals and percents. However, each of these is dealt with from the beginning except for finding the prodcut of two decimals.

3-1. Large Numbers

This section seeks to cultivate the ability to read large numbers, an appreciation for them, and ability to write them in scientific notation. A certain amount of estimation is also included.

Of course there is no largest number and two knowledgable boys would play the game described to a draw by the simple process of adding one, multiplying by two or in some other way increasing the number which their opponent had just given. The British system requires fewer names since it groups the numbers by millions but one must hold his breath longer in saying a number. (Perhaps the British have better lungs.)

Some of the more thoughtful students may wonder why we do not write 93 million, for instance, as 93×10^6 where the exponent is used to indicate the number of zeros in the number. There is no point in trying to hide the fact that in many cases this is really a little simpler, and there is no reason to try to prevent students from using it (see comments on Section 2). But two things should be made clear; in the first place, this is the notation which the scientists use and, second, in the use of logarithms, the scientific notation is certainly much simpler.

Problems 5 and 6 deserve special mention. 9^9 is the product of nine 9's. Now $9^3 = 729$ which is about 700 and $700^3 = 343,000,000$. So 9^9 is about 400 million.

For Problem 6, a very large number would be 9^{9^9}. If this were 10^{9^9} this could be written as a 1 followed by 400 million zeros. (A student might like to estimate how long it would take to write this out.) Hence 9^{9^9} would certainly be larger than a number written as 1 followed by 300 million zeros. This is much larger than any other number mentioned in this unit.

About $1\frac{1}{2}$ days would probably be needed for this section.

Answers to Exercises 3-1.

1. (a) 6.78×10^5 (d) 7.8×10^4
   (b) 9×10^9 (e) 4.59×10^8
   (c) 5×10^3 (f) 7.81×10^9
2. (a) 23,000 (b) 589,700,000 (c) 732,400,000

3. Six hundred seventy eight thousand

   Nine billion

   Five thousand

   Seventy eight thousand

   Four hundred fifty nine million

   Seven billion, eight hundred ten million

   Twenty three thousand

   Five hundred eighty nine billion, seven hundred million

   Seven hundred thirty two million, four hundred thousand

4. 1,395,060

5. 9^9

6. 9^{9^9}

3-2. Calculating with Large Numbers

The objectives of this section are a continuation of those in the first with the added skill of multiplying, using the scientific notation.

Here again, some students may prefer to multiply 93,000,000 by 11,000, for example, by writing the former as 93×10^6 and the latter as 11×10^3, then forming the product: 1023×10^9 and then putting it into scientific notation: 1.023×10^{12}. The teacher may prefer this, too, in which case he or she should use it.

To see that $1023 = 1.023 \times 10^3$ one could refer to the rules for multiplying decimals or the teacher might prefer to go back to first principles. One way to do the latter would be:

$$1.023 = 1 + \frac{23}{1000} \quad \text{and, since} \quad 10^3 = 1000,$$

$$1.023 \times 10^3 = (1 + \frac{23}{1000}) \times 1000 =$$

$$1000 + \frac{23}{1000} \times 1000 = 1000 + 23 = 1023.$$

However, the student should realize that multiplying a number by 10 is equivalent to moving each digit one place to the left, or simply moving the decimal point one place to the right. He should, however, be able to show on demand why either is so.

In Problem 2 notice that the speed of the space ship is a mile per second, which is about 32 million miles a year. The answer to Problem 16, 800,000 years is hard to believe.

Not all of the problems need to be worked, but this section will probably take $2\frac{1}{2}$ days.

Answers to Exercises 3-2.

1. (a) $9 \times 10^9 \times 7 \times 10^4 = 9 \times 7 \times 10^{13} = 63 \times 10^{13} = 6.3 \times 10^{14}$

   (b) $9.3 \times 10^6 \times 7.2 \times 10^4 = 66.96 \times 10^{10} = 6.696 \times 10^{11}$

   (c) $1.25 \times 10^2 \times 1.78 \times 10^{10} = 2.225 \times 10^{12}$

2. 32 million
3. 568
4. About 7
5. About 139 hours
6. 150 days
7. 50

8. 12 days
9. About 30 years, or 11,000 days
10. About 34,722 days or about 95 years
11. 50,000 days or about 10,000 weeks of 5 days each.
12. No. About 2740 years.
13. 500 seconds or about 8 minutes.
14. About 600,000,000 miles. This is about 0.0001 of a light year.
15. About 3 years.
16. 800,000 years.

3-3. Small Numbers

Here small numbers and the scientific notation are dealt with and negative exponents are introduced in a natural way. No definite rule is given for adding exponents because it is felt that this is one rule that the students should gradually develop themselves. Some will get it quickly and others more slowly, but each one should gradually get to the point of formulating this for himself on the basis of experience.

Probably all the exercises should be done. This section should take about $1\frac{1}{2}$ days.

Answers to Exercises 3-3.

1. (a) 10^{-3} (b) 10^{9} (c) 10^{-5} (d) 10^{-11}
2. (a) 16 (b) 125 (c) $\frac{1}{8}$ (d) $\frac{1}{1024}$ (e) 1
   (f) 1 (g) 1

3. (a) 9.3×10^{-2} (b) 7.86×10^{-5}

   (c) 1.57×10^{-1} (d) 1.2356×10^{2}

4. (a) $1 \times 10^{-3} \times 5.7 \times 10^{-2} = 5.7 \times 10^{-5}$

   (b) 2.952×10^{-17}

   (c) 5.472×10^{-12}

   (d) 10

5. $4567 \times 12 \times 10^{-3} = 54.804$ or 5.4804×10

3-4. Products of Large and Small Numbers.

In this section the emphasis is on multiplying large by small numbers. About the chief application of this is to give some meaning to both, since a very large or very small number by itself does not mean much. Again the manipulation of exponents should not be stressed beyond what is needed for the problems at hand and students should do it the long way until they have discovered shorter ways.

The time for this section should be about 1 day.

Answers to Exercises 3-4.

1. (a) 10^{24}

   (b) 10^{-3}

2. (a) 2.6

   (b) 5.52

   (c) $28 \times 10^{-6} = 2.8 \times 10^{-5}$

3. 3.141592653589793

4. $\frac{3}{100}$

5. 6.4×10^{21} is approximately the number of miles in one billion light years. 35×10^{23} is approximately the number of oxygen atoms it takes to weigh one gram. The latter is about 500 times as large as the former.

3-5. Percent

The objective of this section is to show the student some applications of percent and try to get him to the point where he can make applications for himself. If a student understands decimals and that percent is hundredths, not much drill on manipulations with percent should be necessary.

Even though this section will take three days or more, it is not made into several sections because it is very important that no encouragement be given either the teacher or the pupil to classify problems. Such classification would not only obscure the underlying common ideas but make it less likely that the student would discover other applications. It is for this same reason that sample problems of each "type" are not worked out in the text for each kind of problem in the exercises. Furthermore it would be futile to attempt to give examples of all applications which a student might later encounter. For one thing, no one knows what his future will be and for another thing, things will be done differently by the time he is old enough to make application. Try to get him to the point where he will make his own applications, and encourage him to bring in other applications.

Compound interest is touched on. It is not intended that very complicated problems in this be solved, but that merely the fundamental idea be grasped. If one were doing many problems in compound interest, interest tables and the binomial theorem would be necessary. These we leave for some later time. But the idea of interest earning interest is a simple one and should be well within the grasp of such students. Another idea that is important is that in the latter part of the section, namely, that if the rate of interest is 3 percent you can get the amount at the end of the year by miltiplying that at the beginning by 1.03. Perhaps more examples of this could be given. An example of the application of this principle to population increase is also given. This is useful in comparing the answer

$$1000 \times 1.04 \times 1.05$$

to the problem in the text with the answer to Exercise 9 expressed in the form

$$1000 \times 1.05 \times 1.04.$$

Then it is immediately evident by the commutative property for multiplication that the two answers must be the same.

Here the question might be raised: suppose the population increased by the same percent each year, what would it have to be in order that the population at the end of two years be 1092? This could be found exactly by letting r be the rate in hundredths which would accomplish this.

Then $1000 \times (1 + r)$ would stand for the population at the end of the first year and $1000 \times (1 + r)^2$ that at the end of the second. Then

$$1000 \times (1 + r)^2 = 1092$$
$$(1 + r)^2 = 1.092$$
$$1 + r = \sqrt{1.092} .$$

This gives a value of r only slightly less than

$$\frac{1}{2}(0.04 + 0.05)$$

which is 0.045.

For Problem 18, the students might make an estimate from the results in Problem 17. Or the intuition of some bright boy interested in baseball might lead him to the correct result after some experimentation. One way to work this out would be to let k stand for the number of games played by each team with each other team. If the games were listed as in the table for Problem 17, the average for each team would be the number of games listed in its row, divided by k. So the sum of the averages for the ten teams would be the sum of the numbers of all the rows divided by $9k$ since each team plays nine others.

We can get this sum another way. The number in the first row second column plus the number in the second row first column is k; the number in the third row fourth column plus the number in the fourth row third column is also k, and so forth. That is, we can pair entries in this fashion (symmetrical about the diagonal of zeros) so that the sum of the elements in all the rows is k times half the number of entries off the diagonal of zeros. But the number of entries is $10 \times 10 - 10 = 90$ since there are ten entries on the diagonal. Thus the sum of the entries in the table is $45k$ and hence the sum of the averages is $\frac{45k}{9k} = 5$. Notice that the answer is independent of the number of games, k, each team plays with every other team though of course it depends on the condition that each team plays the same number of games with every other team.

But the answer does depend on the number of teams in the league. If R is made to stand for the number of teams in the league, the ratio would be

$$\frac{1}{2} \cdot \frac{(R^2 - R)}{k} \cdot \frac{(R - 1)}{k} = \frac{R}{2}.$$

Thus if R is 10 as in Problem 18 the sum is $10/2 = 5$, as we found; while if R is 4 as in Problem 17, the sum is $4/2 = 2$ as may easily be checked.

This section should take three or four days. The teacher who is accustomed to spending a week on interest, a week on commission, etc., as if they were separate topics is urged to try mixing them up and stressing principles rather than complex computation. Most of the problems involve only simple computation once the principle is understood.

Answers to Exercises 3-5.

1. 12 percent, 3 percent, 153 percent, 0.2 percent
2. 0.15, 1.65, 0.152, 0.002

3. 5 percent
4. \$1,040.
5. \$5.25
6. \$1,250.
7. \$1092.73
8. $(1.015)^2 = 1.030225$ Thus \$1,000 at the end of the year would earn \$30.23 interest instead of \$30.00.
9. See teacher's Commentary.
10. \$136.75
11. \$630.00
12. 23 miles
13. 0.375
14. 0.375
15. 0.750
16. 0.567
17. A: .217 ; B: .450 ; C: .667 ; D: .667.
18. See commentary.
19. \$78.75; \$1.58, \$2.17, 2.9 percent
20. No. See Section 6.
21. No. The price should have been \$25,263.16

3-6. Discount

Discount is considered in a separate section because it is fundamentally different from the other applications in the previous section. The last two problems in section 5 are intended as a preparation for this section. The relationship between discount and interest is important.

Just as in Problem 4, the student may also want to do Problems 5 and 6 relative to certain specific amounts before considering the general situation. Here also the ideas at the close of Section 5 can also be brought into play as follows: In Problem 5, the population of the town at the end of the first year will be its population at the beginning of the year multiplied by $1 - 0.05 = 0.95$ and to get the population at the end of the second year we multiply this by $1 + 0.05 = 1.05$. So the population at the end of the second year can be obtained from that at the beginning by multiplying by

$$0.95 \times 1.05 = 0.9975$$

which amounts to a decrease of 0.25 percent. One could guess a priori that some such result might occur since the 5 percent increase is taken of a smaller amount than the 5 percent decrease. On the other hand, if the problem were changed to have a 5 percent increase first and a 5 percent decrease second, the result would be the same since the product

$$1.05 \times 0.95$$

is equal to the previous one. (In this case the decrease is taken of the larger amount.) Similar remarks apply to Problem 6.

This section should take about $1\frac{1}{2}$ days.

Unit III should take 11 or 12 days.

Answers to Exercises 3-6.

1. $3720
2. 950
3. $25,263.16
4. almost 5.3 percent.
5. $(1 - 0.05)(1 + 0.05) = .9975 = 1 - 0.0025$

   Hence the population at the end is 0.25 percent less than in the beginning.
6. No. $0.75 \times 0.95 = 0.7125$. Thus the discount is 28.75 percent.

A good reference, especially for the first part of this chapter is Gamow, George, "One, Two, Three..Infinity." New York : The Viking Press, 1947.

Unit 4

CONGRUENCE AND THE PYTHAGOREAN PROPERTY

4 - 1 Basic Constructions

Pupils are expected to attain skill in handling the compass and ruler at the same time that they learn to do the basic constructions of geometry. Pupils will need guidance in deciding upon a "convenient" length for the radius. In order to make an accurate construction, intersecting arcs should be made by radii that are relatively short. In Figure 4-1a, the arcs below the line do not need to be made by the same length radii as those above the line. Class discussion can bring out that usually it is simpler to make all four arcs with the same length radius. In Figure 4-1b, pupils may think that it is necessary to make two intersecting arcs on the side of line ℓ opposite from point j. Class discussion can bring out that only two points are needed to construct a line, and one point, p is already given. In the construction of Figure 4-1c it would be interesting to discuss the possibility of having the intersection of arcs from points K and M on the same side of line ℓ as point P. If the two points, P and O are close together it is not easy to draw the line $\overline{PO}$ as accurately as when the two points are farther apart. Notice that the word "congruent" is used to describe segments and angles formerly described as "equal".

Exercise 4-1a Some answers and suggestions.

1. (a) The segment to be bisected is $\overline{JK}$. The rest of the triangle is ignored in the construction.

2. (b) and (c) will need to have the line extended in order to mark arcs on either side of the given point.

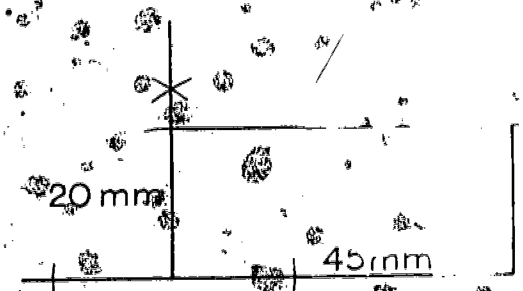
   (f) Lines are parallel

   (g) Each line is parallel to one line and perpendicular to two lines.

3. (d) Line ℓ needs to be extended so that an arc from P will intersect it twice.

   (e) The perpendicular is to be drawn from point J to line . The rest of the triangle is ignored in the construction. It would be interesting to bring out that an altitude of the triangle has been constructed.

*6.



After constructing one right angle, the compass can be used to mark the lengths of 20 mm and 45 mm on the sides of the right angle. The fourth vertex can be located by intersecting arcs drawn from the vertices of the rectangle already located.

The construction of the bisector of an angle is not difficult. Often the same radius is used through-out the construction. The two arcs that intersect in the interior of the angle must be drawn with the same radius. The construction of an angle congruent to a given angle is more difficult. To set the compass accurately for the distance WX is difficult when the compass is a cheap one.

Exercise 4-1b. Some answers and suggestions.

5. (a) Each angle = 45 degrees.
   (b) Each angle < 45 degrees.
   (c) Each angle > 45 degrees and < 90 degrees.

Exercises 4-2a. Some suggestions.

1. The construction uses two angles and a side of triangle JKM. However, the side is not the included side, so the constructed figure is not congruent to triangle JKM.
2. The construction uses two sides and an angle. Since the angle is not the included angle, the constructed figure is not congruent to triangle JKM.
3. The constructed triangle is congruent to triangle JKM, since two angles and the included side of triangle JKM are used.
4. The constructed triangle is congruent to triangle JKM, since two sides and the included angle are used.
5. The constructed triangle is congruent to triangle JKM, since three sides are used.

Exercises 4-2b

1. (a) $\overline{AB} \cong \overline{BC}$, $\overline{AD} \cong \overline{CD}$ Be sure that the pupils understand the word "respectively".
   (b) $\overline{BD} \cong \overline{BD}$ Common side
   (c) Yes Property 3 Pupils should quote the properties, not give the numbers.
   (d) Yes Property 4
2. (a) $\overline{EF} \cong \overline{HJ}$, $\overline{FG} \cong \overline{JK}$, $\overline{EG} \cong \overline{HK}$.
   (b) Yes Property 3
   (c) Yes Property 4
3. (a) Yes $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$
   (b) No There is not a pair of sides equal.
   (c) Triangles would be congruent. Property 2
4. (a) Extends line XZ to make $\overline{ZR} \cong \overline{XZ}$
   (b) Property 1 and Property 4
   (c) Same length as $\overline{QR}$ which can be measured.
5. (a) $\overline{QS} \cong \overline{TV}$, $\overline{QR} \cong \overline{TU}$, $\overline{SR} \cong \overline{VU}$
   (b) $\overline{XY} \cong \overline{CB}$, $\overline{XZ} \cong \overline{AB}$, $\overline{ZY} \cong \overline{CA}$
   (c) $\overline{XZ} \cong \overline{ZR}$, $\overline{ZY} \cong \overline{ZQ}$, $\overline{XY} \cong \overline{QR}$
6. (a) $\angle A \cong \angle E$, $\angle B \cong \angle F$, $\angle C \cong \angle D$.
   (b) $\angle 1 \cong \angle 2$, $\angle X \cong \angle R$, $\angle Y \cong \angle Q$.

*7. CE $\cong$ ED, CF $\cong$ FD, EF $\cong$ EF.

Triangle ECF $\cong$ Triangle EDF by Property 3.

$\angle$ CEO $\cong \angle$ OED by Property 4.

$\overline{EO} \cong \overline{EO}$

Triangle COE $\cong$ triangle OED by Property 1.

$\overline{CO} \cong \overline{OD}$ by Property 4

$\angle 1 \cong \angle 2$ by Property 4

$\angle$ 1 and $\angle$ 2 both measure 90, since together they measure 180.

8. (a) Triangle ECF $\cong$ triangle EDF

Triangle COE $\cong$ triangle OED

Triangle COF $\cong$ triangle DOF

(b) $\angle$ COE $\cong$ $\angle$ DEO

$\angle$ ECO $\cong$ $\angle$ EDO

$\angle$ 1 $\cong$ $\angle$ 2

$\angle$ CFO $\cong$ $\angle$ DFO

$\angle$ FCO $\cong$ $\angle$ FDO

$\angle$ COF $\cong$ $\angle$ DOF

$\angle$ ECF $\cong$ $\angle$ EDF

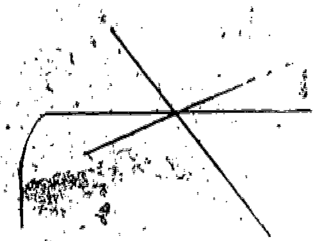
4 - 3 Concurrent Lines

In this section the meaning of concurrent lines is defined, and examples given. The study of concurrent lines is then directed to important applications. The example used concerns the intersection of the perpendicular bisectors of the sides of the triangle. The pupils should be encouraged to make all constructions and drawings carefully. This will aid in giving them an intuitive idea of the facts being discussed throughout this unit. The pupils should be made to appreciate that theorems cannot be proved in the general sense by using special cases or through the use of the straightedge, compass and protractor.

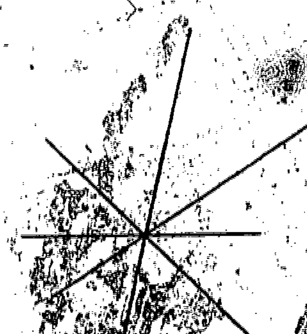
The question raised just following Figure 4-3 is given to indicate that the point of intersection of the perpendicular bisectors of the sides of a triangle is the center of the circle which passes through the vertices of the triangle.

Answers to Exercises 4-3

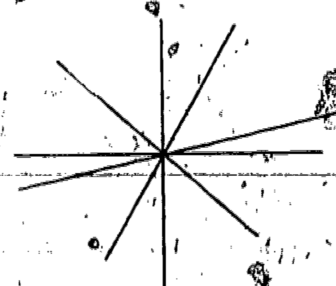
1. (a)

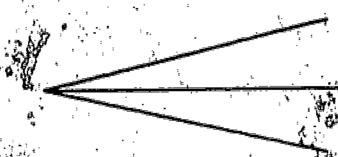


(b)



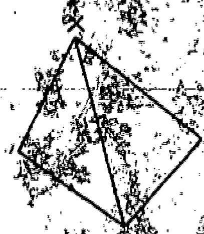
(c)



2. 

3. 2 angles

4. (a) Yes (c) Yes

(b) No (d)  Those intersecting in x and y.

(e) Yes

5. Yes, the medians are concurrent. At this point you may like to introduce the word "centroid".

6. Yes, the altitudes are concurrent.

7. Yes, the bisectors of the angles are concurrent.

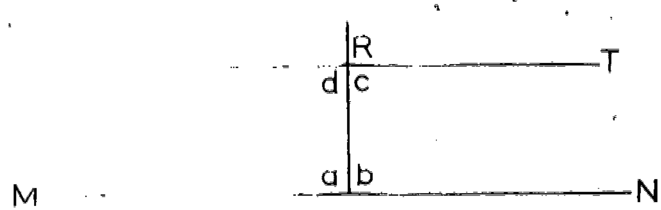
4 - 4 Quadrilaterals.

Although the pupils may know about quadrilaterals, parallelograms, rectangles and squares, it is thought well to review these 4 sided polygons and their relationships: Going a bit beyond what they have previously studied. Before taking up the study of parallelograms a brief study of parallel lines has been introduced. No effort has been made to talk in terms of exterior, interior angles and the like. It is not desirable to use these terms here. The emphasis should be on constructions, drawings and obtaining information concerning the properties of the various configurations.

Only one effort at formal proof is made in this section. This concerns the fact that the diagonals of a parallelogram bisect each other. It is not considered desirable to have much, if any, formal geometrical proofs for eighth grade pupils. In fact, they do not have the mathematical background for this type of work. An attempt at a formal proof once in a while, however, may be of interest, especially to the better students. The proof in this section is not very difficult, so it is recommended that it be studied.

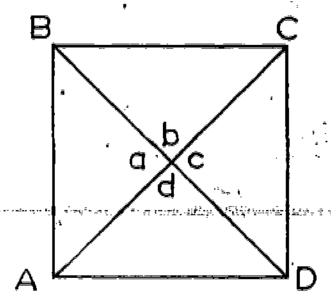
Answers to Exercises 4 - 4

1. (b), (d), and (f).
2. (f)
3. Yes. Because of the relationship of some of the angles formed as discussed in the part on parallel lines.



Angles a, b, c, and d all measure 90. The sum of the measures of angles b and c is 180. Angles b and d are congruent.
4. No. The angles are not right angles.
5. No answer given: A construction problem.

6. Triangle. Also construction.
7. Yes. It is a parallelogram having all angles equal in measure to 90.
8. No. It is a parallelogram having all angles equal in measure to 90, but the sides are not congruent.
9. Yes. A square is a parallelogram, and the diagonals of a parallelogram bisect each other.
10. The intersection angles are all congruent and measure 90.



It can be shown that triangles AOD, DOC, COB, and BOA are all congruent by Property 3, since the diagonals of a square are congruent and bisect each other. Now angles a, b, c and d are congruent by Property 4. The students will realize that $m(\angle a) + m(\angle d) = 180$. It follows then that $m(\angle a)$ is 90, and since the angles are all congruent they all measure 90.

4 - 5 Right Triangle

Because of the interesting properties of the right triangle, especially the Pythagorean Theorem, it is thought desirable to have a section devoted to this topic. The first page gives information concerning the various kinds of triangles.

The remainder of the section deals with the preliminary work leading up to the Pythagorean Theorem and the development of this theorem by use of a special case. The special case is a bit long, by nature, but should prove to be interesting if the pupils are led to do carefully, the construction work and the drawings. At least the better pupils should be able to follow the development without too much difficulty.

Should you feel that the pupils will have too much trouble with the development, you may wish to simply state the theorem, pointing up some of its aspects, and then proceed to the exercise. At the end of this section of the text, there is a table of squares and square roots.

Answers to Exercises 4 - 5
(Some of the Answers)

3. Pythagorean Theorem

4. (a) $25 = 16 + 9$ (c) $625 = 49 + 576$

   (b) $169 = 25 + 144$ (d) $400 = 256 + 144$

6. (a) 2.2 (b) 0.4 (c) 3.6

7. (a) 5 (b) 4.1 (c) 13

8. (a) 2.24 (b) 6.40 (c) 3.61

9. If you know the measure of the area of a square, you find its side by taking the square root of this measure. Measured values of square roots should be approximately the same as computed values.

10. (a) 5.8 (b) 7.8 (c) 9.5 (d) 3.2

11. Square root of $2 \approx 1.4$

12. $\sqrt{3} \approx 1.7$